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Asymmetric cosets

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Abstract: The aim of this work is to present a general theory of coset models G/H in which different left and right actions of H on G are gauged. Our main results include a formula for their modular invariant partition function, the construction of a large set of boundary states and a general description of the corresponding brane geometries. The paper concludes with some explicit applications to the base of the conifold and to the time-dependent Nappi-Witten background.

Keywords: D-branes, Conformal Field Models in String Theory, Conformal and W Symmetry, Boundary Quantum Field Theory.

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Contents

1. Introduction

Many interesting models can be obtained as cosets G/H of a compact group G. Usually, H is identified with a subgroup of G and in forming the coset one employs the adjoint action for which H acts symmetrically from the left and from the right. Such symmetric transformations always possess fixed points (e.g. the group unit). These lead to all kinds of singularities of the resulting coset geometry, including boundaries and corners.

It is possible, however, to work with an enlarged class of exactly solvable cosets and this is the theme of the following note. The idea is to admit different left and right actions of H on G. Even though conformal invariance imposes strong constraints on asymmetric quotients G/H, one gains a lot of freedom in model building. Some of the interesting new theories possess smooth background geometries. One such example is provided by the five-dimensional base $\mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ of the conifold. Other models have isolated singularities such as the big-bang singularity in the four-dimensional Nappi-Witten geometry $\mathrm{SU}(2) \times \mathrm{SL}(2,\mathbb{R})/\mathbb{R} \times \mathbb{R}$.

In spite of these interesting features, asymmetric cosets have not been studied very systematically in the past. One reason for this is that they are typically heterotic, i.e. they possess different left and right chiral algebras. Among the few publications which deal with special cases of asymmetric cosets one may find two early publications by Guadagnini et al. [1, 2]. The models which are studied in these papers can be applied to the base of the conifold as was pointed out some years ago by Pando-Zayas and Tseytlin [3]. Actions for a wider class of asymmetrically gauged WZNW models were written down in [4]. We shall recall below that they are relevant for Nappi-Witten type models [5]. The latter have been employed recently to investigate string theory in time-dependent backgrounds with big-bang singularities [6]. Branes in asymmetrically gauged WZNW models were also studied in [7] but our analysis will give boundary theories with a different geometric interpretation.

The plan of this paper is as follows. In the next section we shall present a comprehensive discussion of the bulk theory and, in particular, spell out an expression for its modular invariant partition function. The third section is then devoted to the construction of boundary states for asymmetric cosets. We will also identify the subspaces along which the corresponding branes are localized. All these results on boundary conditions in asymmetric coset theories are based on our earlier work [8, 9]. In the final section we illustrate the general theory through some important examples. These include the base of the conifold and the Nappi-Witten type coset background.

Note added: while we were preparing this publication, G. Sarkissian issued a paper that has partial overlap with section 4 below [10].

2. The bulk theory

In this first subsection we are going to describe the bulk geometry of asymmetric cosets. We will start with a detailed formulation of the general setup and of the conditions that conformal invariance imposes on the basic data. The origin of the latter can be explained with the help of the classical actions which we shall briefly recall in the second subsection. We then provide expressions for the bulk partition functions and establish their modular invariance. Finally, we present some examples showing the wide applicability of asymmetric cosets. In an appendix to this section we correct some earlier results of Guadagnini et al. [1, 2].

2.1 The geometry of asymmetric cosets

Two groups G and H enter the construction of a coset G/H. Both of them are assumed to be reductive so that they split into a product of simple groups and $\mathrm{U}(1)$ factors. Let the number of these factors be n and r, respectively, i.e. we take G and H to be of the form $G = G_1 \times \cdots \times G_n$ and $H = H_1 \times \cdots \times H_r$. Furthermore, to each factor G_i in the decomposition of G we assign a level k_i. It is convenient to combine the set of all these levels into a vector $k = (k_1, \ldots, k_n)$.

Along with the two groups G and H we need to specify an action of H on G. We take the latter to be of the form $g \mapsto \epsilon_L(h) g \epsilon_R(h^{-1})$ where $\epsilon_{L/R} : H \to G$ denote two group homomorphisms which descend to embeddings of the corresponding Lie algebras. In the usual coset theories ϵ_L and ϵ_R are the same. An asymmetry in the coset construction arises when we drop this condition and allow for two different maps.

The coset space G/H consists of orbits under the action of H on G, i.e.

$$G/H = G/[g \sim \epsilon_L(h) g \epsilon_R(h^{-1})\,;\, h \in H]\,.$$

To be precise, we should display the dependence on the choice of $\epsilon_{L/R}$. But since we consider these maps to be fixed once and for all, we decided to suppress them from our symbol G/H for the coset space. Let us stress, however, that the geometry is very sensitive to the choice of $\epsilon_{L/R}$. We will see this in the examples later on.

The basic data we have introduced so far, i.e. the two groups G, H, the vector k of levels and the maps ϵ_L, ϵ_R, will enter the construction of two-dimensional models with target space G/H. To ensure conformal invariance, however, these data have to obey one important constraint which we can formulate using the notion of an "embedding index" $x_\epsilon \in$Mat$(n \times r)$ for the homomorphism $\epsilon : H \to G$. To define x_ϵ we split ϵ into a matrix of homomorphisms $\epsilon^{si} : H_s \hookrightarrow G_i$ where $s = 1, \ldots, r$, and $i = 1, \ldots, n$, run through the factors of H and G, respectively. The embedding index $x_\epsilon = x = (x^{si})$ is a matrix with elements of the form[1]

$$x^{si} = \frac{\mathrm{Tr}_i\big\{\epsilon^{si}(X)\,\epsilon^{si}(Y)\big\}}{\mathrm{Tr}_s\big\{X\,Y\big\}} \qquad \text{for} \qquad X, Y \in \mathfrak{h}_s \backslash \{0\}\,. \tag{2.1}$$

Observe that the number that is computed by the expression on the right hand side does not depend on the choice of the elements X, Y. Let us also note that the map ϵ^{si} is allowed to map H_s onto the unit element in G_i for some choices of i and s. In this case, the corresponding matrix element x^{si} vanishes.

Let us now consider the embedding indices x_L and x_R for the two homomorphisms ϵ_L and ϵ_R. A conformal theory with target space G/H exists for our choice of levels k, provided that the latter obey the following constraint[2]

$$x_L\, k = x_R\, k\,. \tag{2.2}$$

In other words, the vector of levels must lie in the kernel of $x_L - x_R$. For symmetric cosets this condition is trivially satisfied with any choice of k. Asymmetric cosets, however, constrain the admissible levels.

2.2 The classical action

Using the basic data we have introduced in the previous subsection we can write down the classical action of a gauged WZNW model. As usual, this consists of several pieces. To

[1] We use a normalized trace $\mathrm{Tr} = 2\ \mathrm{tr}\,/I$. Here, we denoted by tr the matrix trace and by I the Dynkin index of the corresponding representation. We use the conventions of [11, pages 58 and 84].

[2] If there are two or more identical groups, this equation has to hold up to a possible relabeling of these groups on one side.

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begin with, there is the WZNW action for the numerator group G,

$$\mathcal{S}^{G}_{\mathrm{WZNW}}(g|k) = \sum_{i=1}^{n} \mathcal{S}^{G_i}_{\mathrm{WZNW}}(g_i|k_i) \tag{2.3}$$

where $g = g_1 \cdots g_n$. This action is a sum over the WZNW actions for the individual groups G_i without any interaction terms. These building blocks are given by

$$\mathcal{S}^{G_i}_{\mathrm{WZNW}}(g_i|k_i) = -\frac{k_i}{4\pi}\int_\Sigma d^2z\ \mathrm{Tr}_i\{\partial g_i g_i^{-1}\,\bar\partial g_i g_i^{-1}\} + \mathcal{S}^{G_i}_{WZ}(g_i|k_i)\,.$$

The Wess-Zumino terms are defined as usual in terms of the Wess-Zumino three-forms ω_i^{WZ}. Consistency of the associated quantum theories enforces quantization constraints on the levels k_i. For simply-connected simple constituents G_i the level k_i has to be an integer. For the U(1) part and non-simply-connected groups the constraints will be different.

The action functional (2.3) is invariant under the "global" transformations of the form $g(z,\bar z) \mapsto g_L(z)\, g(z,\bar z)\, g_R^{-1}(\bar z)$ where $g_L(z)$ and $g_R(\bar z)$ are arbitrary (anti-) holomorphic G-valued functions. Our subgroup H along with the two homomorphisms $\epsilon_{L/R}$ can be used to gauge some part of this WZNW symmetry. To this end we consider the model

$$\mathcal{S}^{G/H}\big(g,A,\bar A\big|\,k,\epsilon_{L/R}\big) = \sum_{i=1}^{n} \mathcal{S}^{G_i}_{\mathrm{WZNW}}(g_i|k_i) + \sum_{i=1}^{n}\sum_{s=1}^{r} \mathcal{S}^{G_i/H_s}_{\mathrm{int}}\big(g_i,A_s,\bar A_s\big|\,k_i,\epsilon^{si}_{L/R}\big)\,. \tag{2.4}$$

Here, the building blocks of the second term are given by [4]

$$\begin{aligned}\mathcal{S}^{G_i/H_s}_{\mathrm{int}}\big(g_i,A_s,\bar A_s\big|\,k_i,\epsilon^{si}_{L/R}\big) = \frac{k_i}{4\pi}\int_\Sigma & d^2z\ \mathrm{Tr}_i\{\,2\,\epsilon_L(\bar A_s)\,\partial g_i g_i^{-1} - 2\,\epsilon_R(A_s)\,g_i^{-1}\bar\partial g_i + \\ & +2\,\epsilon_L(\bar A_s)\,g_i\,\epsilon_R(A_s)\,g_i^{-1} - \epsilon_L(\bar A_s)\,\epsilon_L(A_s) - \epsilon_R(\bar A_s)\,\epsilon_R(A_s)\}\,.\end{aligned} \tag{2.5}$$

In this formula we omitted the superscripts ^{si} on $\epsilon_{L/R}$. The gauge fields $A_s, \bar A_s$ take values in the Lie algebra $\mathfrak{h}_s$. It is not difficult to check that the full action (2.4) is invariant under the following set of infinitesimal gauge transformations

$$\begin{aligned}\delta A_s &= i\,\partial\omega_s + i\,[\omega_s, A_s]\,, \qquad & \delta\bar A_s &= i\,\bar\partial\omega_s + i\,[\omega_s,\bar A_s]\,,\\ \delta g_i &= i\,\epsilon_L(\omega_s) g_i - i\,g_i\epsilon_R(\omega_s) & \text{for } \omega_s &= \omega_s(z,\bar z)\end{aligned}$$

provided that the levels k_i obey the constraint (2.2). In fact, under gauge transformations the action behaves according to

$$\begin{aligned}\delta\mathcal{S}^{G/H}(g|k,\epsilon_{L/R}) = \sum_{i=1}^{n}\sum_{s=1}^{r}\frac{k_i}{4\pi}\int_\Sigma d^2z\ \mathrm{Tr}_i\{&\epsilon^{si}_L(\bar A_s)\,\partial\epsilon^{si}_L(\omega_s) - \epsilon^{si}_R(\bar A_s)\,\partial\epsilon^{si}_R(\omega_s) + \\ &+ \epsilon^{si}_R(A_s)\,\bar\partial\epsilon^{si}_R(\omega_s) - \bar\partial\epsilon^{si}_L(\omega_s)\,\epsilon^{si}_L(A_s)\}\end{aligned}$$

and so it vanishes whenever eq. (2.2) holds true. We have therefore shown that the data introduced above indeed label different two-dimensional conformal field theories.

2.3 Exact solution: modular invariant partition function

Our aim now is to present a few elements of the exact solution. We shall begin with some remarks on the relevant chiral algebras and then address the construction of the modular invariant partition function for our asymmetric coset theories.

In the following let us denote the chiral algebra of the WZNW model for the group G and levels k_i by $\mathcal{A}(G)$. This algebra is generated by a sum of affine Lie algebras with levels k_i, one for each factor in the decomposition of the reductive group G. The two maps $\epsilon_{L/R}$ give rise to two embeddings of the chiral algebra $\mathcal{A}(H)$ into $\mathcal{A}(G)$. Let us note that $\mathcal{A}(H)$ is generated by a sum of affine algebras, one for each factor in the product $H = H_1 \times \cdots \times H_r$. The levels of these affine algebras form a vector $(k'_s)_{s=1,\dots,r}$ whose entries are related to the levels of $\mathcal{A}(G)$ by $k' = x_{L/R}\, k$ (matrix notation). Our assumption (2.2) means that $\epsilon_{L/R}$ give rise to two (possibly different) embeddings of the *same* chiral algebra $\mathcal{A}(H)$ into $\mathcal{A}(G)$. Given these embeddings, we employ the usual GKO construction to obtain two coset algebras $\mathcal{A} = \mathcal{A}(G/H, \epsilon_L)$ and $\bar{\mathcal{A}} = \mathcal{A}(G/H, \epsilon_R)$ which form the left and right chiral algebras of the asymmetric coset model. Note that these two chiral algebras can be different if the two maps ϵ_L and ϵ_R are not the same. In this sense, asymmetric coset models of the kind that we consider in this note are heterotic conformal field theories.

The state space of any conformal field theory decomposes into representations of the chiral algebra. Our task here is to find a combination of these representations which does not only reflect the geometry of the target space G/H but is at the same time also consistent from a conformal field theory point of view. The second requirement means that the vacuum must be unique and that the partition function is modular invariant.

The first condition, namely the relation of our exact solution to the space G/H, implies that in the limit of large levels k the space of ground states has to reproduce the space of functions on G/H. Actually, we can turn this around for a moment and use the harmonic analysis of G/H to get some ideas about the structure of the state space. To this end, let us recall that the algebra $\mathcal{F}(G)$ of functions on G may be considered as a $G \times G$-module under left and right regular action. The Peter-Weyl theorem states that this module decomposes into irreducibles according to

$$\mathcal{F}(G) = \bigoplus V_\mu \otimes V_{\mu^+} ,$$

where μ^+ is the conjugate of μ. Since we want to divide G by the action of H it is convenient to decompose the space of function on G into representations of H. The space of function on G/H is then obtained as the H-invariant part of $\mathcal{F}(G)$. We easily find

$$\begin{aligned} \mathcal{F}(G) &\cong \bigoplus V_\mu \otimes V_{\mu^+} \\ &\cong \bigoplus (b_L)_\mu{}^a \, (b_R)_{\mu^+}{}^{c^+} \, V_a \otimes V_{c^+} \\ &\cong \bigoplus (b_L)_\mu{}^a \, (b_R)_{\mu^+}{}^{c^+} \, N_{ac^+}{}^d \, V_d \, . \end{aligned} \tag{2.6}$$

The symbols $b_{L/R}$ denote the branching coefficients of the inclusion $\epsilon_{L/R}(H) \hookrightarrow G$. The tensor product coefficients $N_{ac^+}{}^d$ for the decomposition of the tensor product of representations of H enter when we restrict the action of $H \times H$ to its diagonal subgroup $H = H_D$.

Taking the invariant part of (2.6) corresponds to putting $d = 0$ or, equivalently, $a = c$ and hence we have shown that

$$\mathcal{F}(G/H) = \mathrm{Inv}_{H_D}\big(\mathcal{F}(G)\big) \cong \bigoplus (b_L)_\mu{}^a\,(b_R)_{\mu^+}{}^{a^+}\,.$$

This is the space that we want to reproduce from the ground states of our exact solution when we send the levels to infinity. With a bit of experience in coset chiral algebras and their representation theory it is not too difficult to come up with a good proposal for the conformal field theory state space that meets this requirement.

The rough idea is to replace the branching coefficients $b_\mu{}^a$ by coset sectors $\mathcal{H}^{G/H}_{(\mu,a)}$. But this rule is a bit too simple and has to be refined in several directions. To build in all the additional subtleties, we need a bit of preparation. For simplicity we consider the sector of the left moving chiral algebra only.

In the following we label sectors of $\mathcal{A}(G)$ by $\mu, \nu, \dots$, and we use the letters $a, b, \dots$, for sectors of $\mathcal{A}(H)$. Let us recall that the two sets of sectors admit an action of the group centers $\mathcal{Z}(G)$ and $\mathcal{Z}(H)$, respectively. This action may be diagonalized by the corresponding modular S-matrices,

$$S^G_{J\mu\,\nu} = e^{2\pi i\,Q_J(\nu)}\,S^G_{\mu\nu} \qquad \text{for } J \in \mathcal{Z}(G)$$

where $Q_J(\nu) = h_J + h_\mu - h_{J\mu}$ are the so-called monodromy charges. An analogous statement holds for the action of the center $\mathcal{Z}(H)$. In a coset sector (μ, a) the labels μ, a form an entity and as such they have to transform identically under the common center

$$\mathcal{G}_{\mathrm{id}}(L) = \big\{(J, J') \in \mathcal{Z}(G) \times \mathcal{Z}(H)\,\big|\, J = \epsilon_L(J')\big\}\,.$$

Not all the labels (μ, a) fulfill this requirement. What remains is the set

$$\mathrm{All}(G/H)_L = \big\{(\mu, a)\,\big|\, Q_J(\mu) = Q_{J'}(a) \text{ for all } (J, J') \in \mathcal{G}_{\mathrm{id}}(L)\big\}$$

of allowed coset labels. It turns out that elements in the set $\mathrm{All}(G/H)_L$ which are related by the action of $\mathcal{G}_{\mathrm{id}}$ correspond to the same coset sector. The set of sectors for the coset chiral algebra is therefore given by $\mathrm{Rep}(G/H)_L = \mathrm{All}(G/H)_L/\mathcal{G}_{\mathrm{id}}(L)$. This observation motivates the term "field identification group" for the common center $\mathcal{G}_{\mathrm{id}}(L)$. The same constructions can be performed for the right chiral algebra. But note that in general the resulting expressions will not coincide.

Having introduced all these notions from the representation theory of coset chiral algebras we are finally able to spell out our proposal for the state space,

$$\mathcal{H}^{G/H} = \bigoplus_{[\mu,a]\in\mathrm{Rep}(G/H)} \mathcal{H}^{(G/H)_L}_{(\mu,a)} \otimes \bar{\mathcal{H}}^{(G/H)_R}_{(\mu,a)^+}\,. \tag{2.7}$$

where the set $\mathrm{Rep}(G/H)$ is defined by

$$\begin{aligned}\mathrm{Rep}(G/H) &= \mathrm{All}(G/H)/\mathcal{G}_{\mathrm{id}} \qquad \text{with}\\ \mathrm{All}(G/H) &= \mathrm{All}(G/H)_L \cap \mathrm{All}(G/H)_R\,, \qquad \mathcal{G}_{\mathrm{id}} = \mathcal{G}_{\mathrm{id}}(L) \cap \mathcal{G}_{\mathrm{id}}(R)\,.\end{aligned} \tag{2.8}$$

Note that the field identification group $\mathcal{G}_{\mathrm{id}}$ admits a natural interpretation as the stabilizer of the action $g \mapsto \epsilon_L(h) g \epsilon_R(h)^{-1}$, i.e.

$$\mathcal{G}_{\mathrm{id}} = \left\{ \left(J, J'\right) \,\middle|\, J' \in \mathcal{Z}(H),\ J = \epsilon_L(J') = \epsilon_R(J') \in \mathcal{Z}(G) \right\} .$$

In writing our formula (2.7) we implicitly assumed that the action of the field identification group $\mathcal{G}_{\mathrm{id}}$ on $\mathrm{All}(G/H)$ possesses no fixed points, i.e. that all orbits $[\mu, a]$ have the same length. It should be stressed that fixed points for the action of $\mathcal{G}_{\mathrm{id}}(L/R)$ on $\mathrm{All}(G/H)_{L/R}$ are not ruled out by this assumption.

As we mentioned before, our proposal (2.7) for the state space has to pass a number of tests before we can accept it as a candidate for the state space of our conformal field theory. From our discussion above it is not difficult to see that at large level, the space of ground states coincides with the space of functions on G/H. Moreover, taking the quotient with respect to $\mathcal{G}_{\mathrm{id}}$ in eq. (2.8) ensures that there is a unique vacuum in $\mathcal{H}^{G/H}$. Hence, it only remains to demonstrate that our Ansatz also leads to a modular invariant partition function. To this end it is convenient to write the partition function in the form

$$Z(q,\bar{q}) = \frac{1}{|\mathcal{G}_{\mathrm{id}}|} \sum_{\mu,a} \mathrm{P}_L^{G/H}(\mu,a)\, \mathrm{P}_R^{G/H}(\mu^+,a^+)\, \chi^{(G/H)_L}_{(\mu,a)}(q) \bar{\chi}^{(G/H)_R}_{(\mu,a)^+}(\bar{q}) .$$

The factor $1/|\mathcal{G}_{\mathrm{id}}|$ in front of this expression removes a common factor from the whole expression in such a way that the vacuum characters possess a trivial prefactor. The summation in the previous expression runs over all labels μ and a and we enforce the restriction to the allowed coset labels by inserting the projectors

$$\mathrm{P}^{G/H}_{L/R}(\mu,a) = \frac{1}{|\mathcal{G}_{\mathrm{id}}(L/R)|} \sum_{(J,J')\in\mathcal{G}_{\mathrm{id}}(L/R)} e^{2\pi i (Q_J(\mu) - Q_{J'}(a))} .$$

It is now rather straightforward to compute how this partition function behaves under the modular transformation S that replaces $q = \exp(2\pi i \tau)$ by $\tilde{q} = \exp(-2\pi i/\tau)$,

$$SZ(q,\bar{q}) = \sum_{\mu,a,\nu,\lambda,b,c} \frac{\mathrm{P}_L^{G/H}(\mu,a)\mathrm{P}_R^{G/H}(\mu^+,a^+)}{|\mathcal{G}_{\mathrm{id}}|}\ S^G_{\mu\nu}\bar{S}^H_{ab}\bar{S}^G_{\mu^+\lambda^+}S^H_{a^+c^+}\ \chi^{(G/H)_L}_{(\nu,b)}(q)\bar{\chi}^{(G/H)_R}_{(\lambda,c)^+}(\bar{q}) .$$

We would like to use unitarity of the S-matrices to simplify this expression. But before we can do so, we have to get rid of the projectors. To this end we insert the explicit formulas for the projectors in terms of monodromy charges and then pull the latter into the S-matrices by shifting their indices with the action of simple currents. This gives

$$\begin{aligned} SZ(q,\bar{q}) = & \frac{1}{|\mathcal{G}_{\mathrm{id}}|\cdot|\mathcal{G}_{\mathrm{id}}(L)|\cdot|\mathcal{G}_{\mathrm{id}}(R)|} \sum_{(J_1,J_1'),(J_2,J_2')} \sum_{\mu,a,\nu,\lambda,b,c} \times \\ & \times S^G_{\mu J_1\nu}\bar{S}^H_{aJ_1'b}\bar{S}^G_{\mu^+J_2\lambda^+}S^H_{a^+J_2'c^+}\ \chi^{(G/H)_L}_{(\nu,b)}(q)\bar{\chi}^{(G/H)_R}_{(\lambda,c)^+}(\bar{q}) . \end{aligned}$$

Now we are able to perform the sum over μ and a to obtain

$$SZ(q,\bar{q}) = \frac{1}{|\mathcal{G}_{\mathrm{id}}|\cdot|\mathcal{G}_{\mathrm{id}}(L)|\cdot|\mathcal{G}_{\mathrm{id}}(R)|} \sum_{(J_1,J_1'),(J_2,J_2')} \sum_{\nu,\lambda,b,c} \delta^{J_2^{-1}\lambda}_{J_1\nu}\delta^{J_1'b}_{J_2'^{-1}c}\, \chi^{(G/H)_L}_{(\nu,b)}(q)\bar{\chi}^{(G/H)_R}_{(\lambda,c)^+}(\bar{q}) .$$

At this stage we may resum the label. Then we see that part of the prefactor cancels and we are left with

$$SZ(q,\bar{q}) = \frac{1}{|\mathcal{G}_{\rm id}|} \sum_{\nu,b} \chi^{(G/H)_L}_{(\nu,b)}(q)\bar{\chi}^{(G/H)_R}_{(\nu,b)^+}(\bar{q})\,.$$

This is exactly the behavior modular invariance requires from our partition function. Note that the restriction to allowed coset labels is implicitly contained in the previous expression since coset characters vanish if the relevant branching selection rule is not satisfied. It is obvious that our partition function is also invariant under modular T-transformations which send τ to $\tau+1$.

2.4 Special cases and examples

Our general construction includes a number of interesting special cases. The most familiar examples are the Nappi-Witten background [5] and the T^{pq}-spaces [3]. Both of them belong to a distinguished class of asymmetric cosets for which we introduce the notion of "generalized automorphism type". In the last subsection we will briefly discuss one example of an asymmetric coset that is not of this type.

2.4.1 Asymmetric cosets from automorphisms

The simplest setup for asymmetric cosets that one can imagine is one in which the left and right embeddings are related by automorphisms. More precisely, we are thinking of situations in which the left homomorphism $\epsilon_L = \epsilon$ is related to $\epsilon_R = \Omega_G^{-1} \circ \epsilon \circ \Omega_H$ by composition with two automorphisms Ω_G and Ω_H of G and H, respectively. Let us notice that the concatenation of an embedding with an automorphism gives another embedding with the same embedding index.[3] This observation guarantees the validity of the anomaly cancellation condition (2.2).

For the explicit construction of the state space (2.7) we have to know the centers $\mathcal{G}_{\rm id}(L)$ and $\mathcal{G}_{\rm id}(R)$ in detail. Note that every element $\big(\epsilon(J'),J'\big) \in \mathcal{G}_{\rm id}(L)$ is mapped to an element $\big(\Omega_G^{-1}\circ\epsilon(J'),\Omega_H^{-1}(J')\big) = \big(\Omega_G^{-1}\circ\epsilon\circ\Omega_H(\Omega_H^{-1}(J')),\Omega_H^{-1}(J')\big) \in \mathcal{G}_{\rm id}(R)$ by the action of the pair $(\Omega_G^{-1},\Omega_H^{-1})$. The right center is thus the image of the left center, $\mathcal{G}_{\rm id}(R) = (\Omega_G^{-1},\Omega_H^{-1})\big(\mathcal{G}_{\rm id}(L)\big)$, and the common center is the intersection of these two sets. Similarly the allowed coset labels are related by $\mathrm{All}(G/H)_R = (\Omega_G^{-1},\Omega_H^{-1})\big(\mathrm{All}(G/H)_L\big)$. To prove this statement one employs the invariance property $Q_{\Omega_G(J)}\big(\Omega_G(\mu)\big) = Q_J(\mu)$ of the monodromy charges and the analogous statement for the subgroup H.

These observations enable us to find a rather explicit expression for the state space. In our example the general formula (2.7) can be simplified due to the fact that left and right moving chiral algebra are isomorphic. We will therefore express the state space in terms of quantities of the left chiral algebra. All we need to do is to replace the coset representations $\mathcal{H}^{(G/H)_R}_{(\mu,a)}$ through $\mathcal{H}^{(G/H)_L}_{(\Omega_G(\mu),\Omega_H(a))}$. By construction, the latter is non-trivial if and only if the first one was. We can also express the action of the common center on

[3]In our terminology, automorphisms do not only have to respect the group multiplication but also the Killing form (or an other invariant form in terms of which the model is defined).

these labels. If we combine these facts we finally arrive at

$$\mathcal{H}^{G/H} = \bigoplus_{[\mu,a]\in\mathrm{Rep}(G/H)} \mathcal{H}^{(G/H)_L}_{(\mu,a)} \otimes \bar{\mathcal{H}}^{(G/H)_L}_{(\Omega_G(\mu),\Omega_H(a))^+} \, .$$

Let us emphasize once more that the coset sectors are *both* defined with respect to the *same* embedding ϵ in this expression. The asymmetry enters in the explicit appearance of the twists of labels and in an (implicit) reduction of labels over which we sum.

The most prominent example of asymmetric cosets of the type considered in this subsection is provided by the Nappi-Witten background [5]. It is obtained as a coset of the product group $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ with respect to some abelian subgroup $H = \mathbb{R} \times \mathbb{R}$. In this case, the automorphism Ω_G is trivial while Ω_H exchanges the two factors of $\mathbb{R}$. The model will be discussed in detail in section 4.

2.4.2 Examples of GMM-type

Let us now consider a slightly more complicated family of examples in which the numerator group is a product $G_1 \times G_2$ of two groups G_1 and G_2 which possess a common subgroup H. Our aim is to describe the coset $G_1 \times G_2/H$ where the first homomorphism $\epsilon_L = e \times \epsilon_2$ embeds H into the group G_2 and $\epsilon_R = \epsilon_1 \times e$ sends elements of H into G_1. The lagrangian description of such models was developed by Guadagnini, Martellini and Mintchev (GMM) more than fifteen years ago [1, 2]. In appendix A we show how their results can be recovered from the more general expression (2.4). We also use the opportunity to correct some statements of GMM concerning the current algebra relations and the validity of the affine Sugawara / coset construction for this type of coset models.

The lagrangian treatment of appendix A and algebraic intuition lets us suspect that the coset model is manifestly heterotic with chiral algebras given by

$$\mathcal{A}\Big((G_1)_{k_1} \otimes \big((G_2)_{k_2}/H_k\big)\Big) \otimes \overline{\mathcal{A}\Big(\big((G_1)_{k_1}/H_k\big) \otimes (G_2)_{k_2}\Big)} \, .$$

One can easily see that the field identification group for the coset $(G_1 \times G_2)/H$ is given by

$$\mathcal{G}_{\mathrm{id}} = \big\{(0,0,J') \,\big|\, (0,J') \in \mathcal{G}_{\mathrm{id}}(G_1/H) \cap \mathcal{G}_{\mathrm{id}}(G_2/H)\big\} \, .$$

The allowed coset labels consist of triples (μ,α,a) such that (μ,a) and (α,a) are allowed for the G_1/H and G_2/H cosets, respectively. Coset representations are then obtained by dividing out the field identifications $\mathcal{G}_{\mathrm{id}}$. The resulting state space simply reads

$$\mathcal{H} = \bigoplus_{[\mu,\alpha,a]\in\mathrm{Rep}(G/H)} \mathcal{H}^{G_1}_{\mu} \otimes \mathcal{H}^{G_2/H}_{(\alpha,a)} \otimes \bar{\mathcal{H}}^{G_1/H}_{(\mu,a)^+} \otimes \bar{\mathcal{H}}^{G_2}_{\alpha^+} \, .$$

It reflects the fact that in both the left and the right moving algebra one still finds a residual current symmetry.

For the physical applications we are particularly interested in a special choice of product group and subgroup, $G_1 = G_2 = \mathrm{SU}(2)$ and $H = \mathrm{U}(1)$. Under these circumstances the GMM-model describes five-dimensional non-Einstein T^{pq} spaces [3]. The special case $p = q = 1$ admits a direct interpretation as the base of the conifold (see, e.g., [12]). This example will be discussed in detail in section 4.

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2.4.3 Asymmetric cosets of non-automorphism type

In the last two subsections we discussed examples of asymmetric cosets which are of rather special form. Recall that for the first case, the left and right embeddings were simply related by automorphisms. An interesting generalization of this setup involves choosing a chain of subgroups $H = U_1 \subset U_2 \subset \cdots \subset U_N = G$ along with left and right embeddings which are pairwise related through automorphisms. If an asymmetric coset falls into this wider class, we will say that it is of "generalized automorphism type". Note that the GMM coset models belong to this family. To see how this works, let us introduce a subgroup $U_2 = H \times H$ which sits in between H and $G = G_1 \times G_2$. Given such an intermediate group we first embed H into either the second or first factor of $H \times H$ and then continue by embedding $H \times H$ into G. In this scenario, the left and right embeddings from H to $H \times H$ are related by the permutation automorphism of $H \times H$ and the left and right embedding from $H \times H$ to G are even identical.

Asymmetric coset models of generalized automorphism type are heterotic with respect to their maximal chiral algebras, i.e. the algebra of holomorphic chiral fields is not isomorphic to the algebra of anti-holomorphic fields (unless the model is of automorphism type). On the other hand, their chiral algebras possess a smaller common chiral subalgebra for which the whole theory is still rational. This property will enable us in the next section to write down a large number of boundary states for asymmetric cosets of generalized automorphism type.

Before we proceed to the discussion of boundary conditions, however, we would like to provide at least one example of an asymmetric coset that is not of generalized automorphism type. To this end we consider once again the product $G = G_1 \times G_2$ of two simple Lie groups G_i which possess a common subgroup H. One can define an action of H on G which is based on the embeddings of the following special form $\epsilon_L(h) = \big(\epsilon_1(h), \epsilon_2(h)\big)$ and $\epsilon_R(h) = \big(\epsilon_1'(h), \mathrm{id}\big)$. The corresponding matrices of embedding indices are denoted by (x_1, x_2) and $(x_1', 0)$. If we can now find levels such that the condition $k = x_1 k_1 + x_2 k_2 = x_1' k_1$ is obeyed, then there exists an associated asymmetric coset model with chiral algebra

$$\mathcal{A}\Big(\big((G_1)_{k_1} \times (G_2)_{k_2}\big)/H_k\Big) \otimes \overline{\mathcal{A}\Big(\big((G_1)_{k_1}/H_k\big) \times (G_2)_{k_2}\Big)}\,.$$

For models of this type we were not able to find a common chiral subalgebra for which the theory stays rational. An very explicit example is obtained using the inclusion $\mathfrak{su}(2)_{4k'} \hookrightarrow \mathfrak{su}(3)_{k\prime} \oplus \mathfrak{su}(3)_{3k'}$. In fact, there are two embeddings of $\mathfrak{su}(2)$ into $\mathfrak{su}(3)$ at our disposal with embedding indices 1 and 4, respectively. If we employ the embedding with index 1 for ϵ_i and choose ϵ_1' such that it has embedding index 4, then the anomaly cancellation condition is satisfied.

3. The boundary theory

In this section we will construct boundary states for the asymmetric coset theories. The heterotic nature of these models will force us to break part of the bulk symmetry. But for a large class of asymmetric cosets we will be able to identify smaller chiral symmetries for which the boundary theory remains rational. Our discussion starts with a short reminder

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on maximally symmetric and symmetry breaking branes on group manifolds. We will then argue that some of the symmetry breaking branes on G can descend to the asymmetric coset and we will identify the localization of these branes in G/H. Formulas for the boundary states and the partition functions of the boundary theories will be provided at the end of the section.

3.1 Branes on group manifolds

Among the branes on group manifolds, maximally symmetric branes are distinguished since they preserve the whole chiral current algebra symmetry. The construction of maximally symmetric boundary conditions in the WZNW model requires to choose some gluing automorphism Ω of the chiral algebra $\mathcal{A}(G)$ so that we can glue holomorphic and anti-holomorphic currents along the boundary. Before we describe a few results from boundary conformal field theory of the corresponding branes, let us briefly look at the geometric scenario these boundary conditions are associated with. It is by now well known that branes constructed with $\Omega = \mathrm{id}$ are localized along conjugacy classes [13]. The general case has an equally simple and elegant interpretation [14]. Note that gluing automorphisms Ω for the current algebra $\mathcal{A}(G)$ are associated with automorphisms of the finite dimensional Lie algebra $\mathfrak{g}$ which, after exponentiation, give rise to an automorphism Ω^G of the group G. One can then show that maximally symmetric branes are localized along the following *twisted conjugacy classes* in the group manifold,

$$\mathcal{C}^\Omega_u := \left\{ g\, u\, \Omega^G(g^{-1}) \;\middle|\; g \in G \right\}.$$

The subsets $\mathcal{C}^\Omega_u \subset G$ are parametrized through equivalence classes of group elements u where the equivalence relation between two elements $u, v \in G$ is given by: $u \sim_\Omega v$ iff $v \in \mathcal{C}^\Omega_u$. One should think of u as a coordinate that describes the transverse position of the brane on the group manifold. In the exact conformal field theory, these coordinates are quantized.

The algebraic description of maximally symmetric D-branes was developed in [15] (see also [16]). Their boundary states are labeled by representations of the twisted Kac-Moody algebra which may be constructed from the Lie algebra $\mathfrak{g}$ using the automorphism Ω. They are specific linear combinations of certain generalized coherent (or Ishibashi) states,

$$|u\rangle = \sum_{\Omega(\mu)=\mu} \frac{\psi_u{}^\mu}{\sqrt{S_{0\mu}}}\, |\mu\rangle\rangle \,.$$

As usual, the generalized coherent states only implement the gluing conditions for the currents and there is one such state for each Ω-symmetric combination of irreducible $\hat{\mathfrak{g}}$-representations in the charge conjugate state space of the WZNW theory. The coefficients $\psi_u{}^\mu$ in the previous formula are directly related to the one-point functions of bulk fields in the boundary theories and explicit expressions can be found in the literature [15]. From the boundary states one can compute the partition function

$$Z_{uv} = \sum_{\nu \in \mathrm{Rep}(G)} \left(n_\nu\right)_v{}^u \chi_\nu = \sum_{\nu \in \mathrm{Rep}(G)} \sum_{\mu = \Omega(\mu)} \frac{\bar{\psi}_u{}^\mu\, \psi_v{}^\mu\, S_{\nu\mu}}{S_{0\mu}}\, \chi_\nu$$

for each pair of labels u, v. The numbers $\left(n_\nu\right)_v{}^u \in \mathbb{N}_0$ are the twisted fusion rules of $\hat{\mathfrak{g}}$. For details of the construction we refer the reader to the existing literature.

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In addition to these maximally symmetric branes, a large class of symmetry breaking branes has been obtained in [8]. Their geometry was identified later in [9]. The construction of these branes requires to choose a chain of groups $U_s, s = 1, \ldots, N$, along with homomorphisms $\epsilon_s : U_s \to U_{s+1}$ (we set $U_N = G$). The latter are again assumed to induce embeddings of the corresponding Lie algebras. Furthermore, one has to select an automorphism Ω_s on each group U_s. Given these data, it is possible to construct a set of branes which preserve an U_1 group symmetry. These are localized along the following sets

$$\begin{aligned} \mathcal{C}^{\underline{\Omega}}_{\underline{\epsilon};\underline{u}} &= \mathcal{C}^N_{u_N} \cdot \mathcal{C}^{N-1}_{u_{N-1}} \cdot \cdots \cdot \mathcal{C}^1_{u_1} \subset G \qquad \text{where} \\ \mathcal{C}^s_{u_s} &= \Omega_N \circ \epsilon_{N-1} \circ \cdots \circ \Omega_{s+1} \circ \epsilon_s(\mathcal{C}^{\Omega_s}_{u_s}) \subset G \qquad \text{for } u_s \in U_s \end{aligned} \tag{3.1}$$

and $\mathcal{C}^N_{u_N} = \mathcal{C}^{\Omega_N}_{u_N}$ for $u_N \in G$. The $\cdot$ indicates that we consider the set of all points in G which can be written as products (with group multiplication) of elements from the various subsets. One should stress that branes may be folded onto the subsets (3.1) such that a given point is covered several times. This phenomenon has been observed for a special case in [17]. For maximally symmetric branes on ordinary adjoint cosets the previous geometry reduces to simpler expressions which have been found before [18, 19].

To illustrate this abstract construction, we show how to recover the symmetry breaking branes on SU(2) that were found in [17]. In this case, we choose a chain of length $N = 2$ and set $U_1 = \mathrm{U}(1)$. Let us then fix the automorphism Ω_1 on U(1) to be the inversion $\Omega_1(\eta) = \eta^{-1}$ for all $\eta \in \mathrm{U}(1)$. The automorphism Ω_2 of SU(2) is assumed to be trivial and ϵ_1 can be any embedding of U(1) into SU(2). With these choices, the twisted conjugacy classes $\mathcal{C}^{\Omega_1}$ fill the whole one-dimensional circle U_1. When we multiply points of the circle with elements in the spherical conjugacy classes $\mathcal{C}^{\Omega_2} = \mathcal{C}^{\mathrm{id}}$ of SU(2) the resulting set sweeps out a three-dimensional subspace of SU(2) which can degenerate to a 1-dimensional circle. Hence, for this very special example, we reproduce exactly the findings of [17].

3.2 Branes in asymmetric cosets

Having constructed maximally symmetric and symmetry breaking branes in the group G, our strategy now is to investigate which of these branes can pass down to the asymmetric coset G/H. Geometrically, this is not too hard to understand. In fact, the natural idea is to look at all the symmetry breaking branes which are obtained from chains starting with $U_1 = H$ and end at $U_N = G$ and to impose an extra condition on the choice of the automorphisms Ω_s and the homomorphisms ϵ_s so as to reflect the action of H on G in the coset construction. Explicitly, the conditions on Ω_s and ϵ_s read

$$\epsilon_L = \epsilon_{N-1} \circ \cdots \circ \epsilon_1 \,, \tag{3.2}$$

and

$$\epsilon_R = \Omega^{U_N} \circ \epsilon_{N-1} \circ \Omega^{U_{N-1}} \circ \cdots \circ \Omega^{U_2} \circ \epsilon_1 \circ \Omega^{U_1} \,, \tag{3.3}$$

Our claim is that the subsets (3.1) that are obtained from chains (U_s, Ω_s) with homomorphisms ϵ_s pass down to subsets on the asymmetric coset G/H, provided that the data of the chain are related to the data $\epsilon_{L/R}$ of the asymmetric coset G/H by eqs. (3.2) and (3.3). We believe that the branes that are obtained in this way are the only ones that possess a rational boundary theory.

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3.3 Boundary states and partition function

We now turn to the exact solution of the boundary conformal field theories which are used to describe the branes we talked about in the previous subsection. Our assumptions on the existence of a chain of embeddings and its properties guarantee that the resulting theories are rational with respect to a chiral symmetry

$$\mathcal{A} = \mathcal{A}(U_N/U_{N-1}) \otimes \cdots \otimes \mathcal{A}(U_3/U_2) \otimes \mathcal{A}(U_2/U_1)\,.$$

Let us stress that the left and right chiral algebra are isomorphic after symmetry reduction while this did not have to be the case before. For the rest of this section, let us restrict to embedding chains of length $N = 3$. This does not only cover all the examples to be discussed later on, but it also simplifies our notations. The extension to the general case will be straightforward. We will also set $\Omega_1 = \mathrm{id} = \Omega_3$ and write $U_2 = U$.

To begin with, it is convenient to rewrite the bulk partition function of the asymmetric coset model in terms of characters for the chiral algebra $\mathcal{A}$ that is left unbroken by the boundary condition. With our simplifying assumptions, this partition function becomes

$$\mathcal{H} = \bigoplus_{[\mu,a]\in\mathrm{Rep}(G/H)} \bigoplus_{\alpha,\beta\in\mathrm{Rep}(U)} \mathcal{H}^{G/U}_{(\mu,\alpha)} \otimes \mathcal{H}^{U/H}_{(\alpha,a)} \otimes \bar{\mathcal{H}}^{G/U}_{(\mu,\beta)^+} \otimes \bar{\mathcal{H}}^{U/H}_{(\Omega(\beta),a)^+}\,. \tag{3.4}$$

We had to include the automorphisms Ω in one of the coset representations because in the original formulation of the symmetry reduction left and right chiral algebra are just isomorphic, not identical. By explicit insertion of Ω we are able to formulate the theory in terms of one single chiral algebra $\mathcal{A}$.

To construct boundary states we have to find the symmetric part of the Hilbert space (3.4). From the G/U cosets we obtain the condition $\alpha \equiv \beta$ modulo field identification of the form $(e, J) \in \mathcal{G}_{\mathrm{id}}(G/U)$. From the U/H cosets one arrives at $\alpha = \Omega(\beta)$. This is due to the fact that elements of the field identification group $\mathcal{G}_{\mathrm{id}}(U/H)$ can not have the form (J, e). The first condition then translates into $\alpha = J\Omega(\alpha)$. We will assume that this condition can only be fulfilled for $\Omega(\alpha) = \alpha$.[4] Generalized coherent states $|\mu, \alpha, a\rangle\rangle$ for this setup are labeled by triples μ, α, a such that

$$\begin{aligned} &(\mu, \alpha) \in \mathrm{All}(G/U)\,, \qquad (\alpha, a) \in \mathrm{All}(U/H)\,,\\ &\Omega(\alpha) = \alpha\,. \end{aligned}$$

In addition we have to identify these generalized coherent states according to the identification rule

$$|J\mu, \alpha, J'a\rangle\rangle \sim |\mu, \alpha, a\rangle\rangle \qquad \text{for } (J, J') \in \mathcal{G}_{\mathrm{id}}\,.$$

Let ${\psi_z}^{\alpha}$ be the structure constants of twisted D-branes in the target space U. Then we may define boundary states for the asymmetric coset by

$$|\rho, z, r\rangle = \sum \mathrm{P}(\mu, \alpha)\mathrm{P}(\alpha, a)\ \frac{S^G_{\rho\mu}}{\sqrt{S^G_{0\mu}}}\,\frac{{\psi_z}^{\alpha}}{S^U_{0\alpha}}\,\frac{\bar{S}^H_{ra}}{\sqrt{S^H_{0a}}}\,|\mu, \alpha, a\rangle\rangle\,.$$

[4] This condition can be non-trivial only if there exist elements in the center of H which are mapped to the unit element by both ϵ_L and ϵ_R.

We note that this is a consistent prescription since the formula does not depend on the specific representative of the Ishibashi states. Also, we may implement the identification of boundary states

$$|J\rho, z, J'r\rangle \sim |\rho, z, r\rangle \qquad \text{for } (J, J') \in \mathcal{G}_{\text{id}}\,.$$

Using world-sheet duality it is not difficult to derive formulas for the boundary partition functions. As usual we start from the following expression involving the coefficients of boundary states

$$Z = \sum \mathrm{P}(\mu,\alpha)\mathrm{P}(\alpha,a)\,\frac{\bar{S}^G_{\rho_1\mu}S^G_{\rho_2\mu}}{S^G_{0\mu}}\,\frac{\bar{\psi}^{\alpha}_{z_1}\psi_{z_2}{}^{\alpha}}{S^U_{0\alpha}S^U_{0\alpha}}\,\frac{S^H_{r_1a}\bar{S}^H_{r_2a}}{S^H_{0a}}\,\chi^{G/U}_{(\mu,\alpha)}\chi^{U/H}_{(\alpha,a)}(\tilde{q})$$

and perform the modular S transformation to obtain

$$Z = \sum \mathrm{P}(\mu,\alpha)\mathrm{P}(\alpha,a)\,\frac{\bar{S}^G_{\rho_1\mu}S^G_{\rho_2\mu}S^G_{\nu\mu}}{S^G_{0\mu}}\,\frac{\bar{\psi}^{\alpha}_{z_1}\psi_{z_2}{}^{\alpha}\bar{S}^U_{\beta\alpha}S^U_{\gamma\alpha}}{S^U_{0\alpha}S^U_{0\alpha}}\,\frac{S^H_{r_1a}\bar{S}^H_{r_2a}\bar{S}^H_{ba}}{S^H_{0a}}\,\chi^{G/U}_{(\nu,\beta)}\chi^{U/H}_{(\gamma,b)}(q)\,.$$

We now want to pass to an unrestricted sum over μ, α, a (α still has to be symmetric). This can be achieved if we express the projectors in terms monodromy charges and pull the corresponding simple currents into the S matrices. We are thus lead to

$$Z = \frac{1}{|\mathcal{G}^{G/U}_{\text{id}}|\,|\mathcal{G}^{U/H}_{\text{id}}|}\sum \frac{\bar{S}^G_{\rho_1\mu}S^G_{\rho_2\mu}S^G_{J_1\nu\mu}}{S^G_{0\mu}}\,\frac{\bar{\psi}^{\alpha}_{z_1}\psi_{z_2}{}^{\alpha}\bar{S}^U_{J'_1\beta\alpha}S^U_{J_2\gamma\alpha}}{S^U_{0\alpha}S^U_{0\alpha}}\,\frac{S^H_{r_1a}\bar{S}^H_{r_2a}\bar{S}^H_{J'_2ba}}{S^H_{0a}}\,\chi^{G/U}_{(\nu,\beta)}\chi^{U/H}_{(\gamma,b)}(q)\,.$$

The expression may be evaluated directly by means of the Verlinde formula. The final result is

$$\begin{aligned} Z &= \frac{1}{|\mathcal{G}^{G/U}_{\text{id}}|\,|\mathcal{G}^{U/H}_{\text{id}}|}\sum\; N_{\rho_1^+,\rho_2,J_1\nu}N^{\delta}_{(J'_1\beta)^+,J_2\gamma}\big(n_\delta\big)_{z_1}{}^{z_2}N^{J'_2b}_{r_1r_2^+}\,\chi^{G/U}_{(\nu,\beta)}\chi^{U/H}_{(\gamma,b)}(q)\\ &= \sum\; N_{\rho_1^+,\rho_2,\nu}N^{\delta}_{\beta+\gamma}\big(n_\delta\big)_{z_1}{}^{z_2}N^{b}_{r_1r_2^+}\,\chi^{G/U}_{(\nu,\beta)}\chi^{U/H}_{(\gamma,b)}(q)\,. \end{aligned}$$

Let us remark that this spectrum is consistent with the proposed geometric interpretation. The relevant computations are left to the reader (see [9] for a closely related analysis).

4. Examples

To illustrate the abstract formulas we presented in this work we will now study three important examples. Our discussion starts with a short analysis of D-branes in the parafermionic cosets SU(2)/ U(1) and branes therein. We then proceed to the spaces T^{pq} generalizing the base of the conifold. The section concludes with a detailed investigation of branes in the Nappi-Witten background.

4.1 Branes in the parafermion background

Parafermion theories arise from cosets of the form SU(2)/ U(1). There exist two choices of how the U(1) subgroup can be gauged: adjoint (vectorial) and axial gauging. D-branes for these models have been worked out in [17, 7] and we will not have anything new to

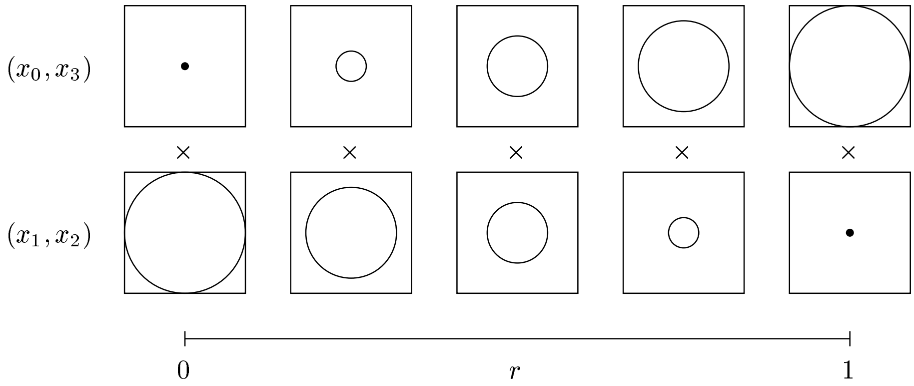

Figure 1: The group manifold SU(2) as a fibre over the unit intervall.

see in this example. Our purpose is merely to introduce some of the tools that help to illustrate the bulk and brane geometries in specific examples. Recovering the geometry of the so-called A and B branes in parafermion theories from our general theory will be rather easy.

The first ingredient in our discussion is the SU(2) group manifold itself. It will be useful for us to parametrize it in terms of two complex coordinates z_1, z_2,

$$g = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \qquad \text{with } \; z_1, z_2 \in \mathbb{C} \; \text{ and } \; |z_1|^2 + |z_2|^2 = 1\,.$$

To picture this space, we define the quantity $r = |z_1|$ which takes values in the interval $0 \le r \le 1$. Over each point $r \in [0,1]$ the group manifold fibers into the direct product $S^1_r \times S^1_{\sqrt{1-r^2}}$ of two circles with radii r and $\sqrt{1-r^2}$, respectively. Hence, we find

$$\mathrm{SU}(2) = \big\{ (z_1, z_2) \,\big|\, |z_1|^2 + |z_2|^2 = 1 \big\} = \big\{ \big(r\, e^{i\phi_1}, \sqrt{1-r^2}\, e^{i\phi_2}\big) \big| 0 \le r \le 1 \big\} \subset \mathbb{C}^2\,.$$

If we identify the complex numbers with euclidean 2-planes according to $z_1 = x_0 + ix_3$ and $z_2 = x_1 + ix_2$ we arrive at the figures 1 and 2.

Next we turn to the U(1) subgroup and its embeddings into SU(2). For the left embedding ϵ_L we shall use the map

$$\epsilon_L : \quad e^{i\tau} \;\mapsto\; \big(e^{i\tau}, 0\big)\,.$$

The vector and axial gaugings arise from two different choices of the right homomorphism ϵ_L which we choose to be

$$\epsilon_R^{v/a} : \quad e^{i\tau} \;\mapsto\; \big(e^{\pm i\tau}, 0\big)\,.$$

The associated actions of U(1) on SU(2) assume a rather simple form in the coordinates (z_1, z_2),

$$\big(z_1, z_2\big) \mapsto \big(z_1, \, e^{2i\tau} z_2\big) \quad \text{(vector)} \qquad \text{and} \qquad \big(z_1, z_2\big) \mapsto \big(e^{2i\tau} z_1, z_2\big) \quad \text{(axial)}\,.$$

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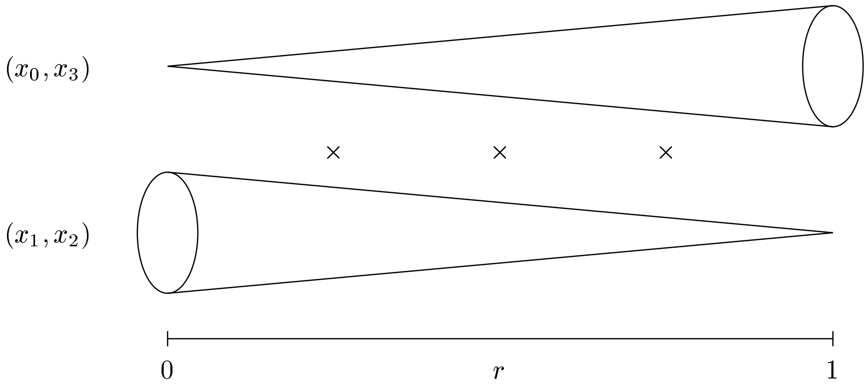

Figure 2: A second illustration of the group manifold SU(2).

In descending to the coset geometry, it is convenient to fix the gauge such that either z_1 or z_2 is a positive real number. We thus arrive at the expressions

$$\mathrm{SU}(2)/\,\mathrm{U}(1)_{\text{vector}} = \big\{\big(r\,e^{i\phi}, \sqrt{1-r^2}\big)\,\big|\, 0 \le r \le 1\big\} \cong D^2$$
$$\mathrm{SU}(2)/\,\mathrm{U}(1)_{\text{axial}} = \big\{\big(r, \sqrt{1-r^2}\,e^{i\phi}\big)\,\big|\, 0 \le r \le 1\big\} \cong D^2\,.$$

In both cases the target space is topologically given by a disc D^2. This can also easily be inferred from the figures 1 and 2.

Let us now proceed to the geometry of branes in this geometry. Our general recipe instructs us to search for embedding chains of some depth N and then to pick automorphisms Ω_s for each of the groups in the chain. Here our chains will have length $N = 2$, they consist of $U_1 = \mathrm{U}(1)$ and $U_2 = \mathrm{SU}(2)$ with some homomorphism $\epsilon : U_1 \to \mathrm{SU}(2)$. On $U_1 = \mathrm{U}(1)$ there exist two different automorphisms Ω_1, namely the identity id and the inversion γ. The latter sends each $\eta \in \mathrm{U}(1)$ to its inverse $\gamma(\eta) = \eta^{-1}$. Automorphisms Ω_2 of SU(2) are all inner so that they are parametrized by elements of SU(2). As we discussed above, these data have to obey the two conditions (3.2), (3.3). In our situation this means that $\epsilon = \epsilon_L$ and $\Omega_2 \circ \epsilon \circ \Omega_1 = \epsilon_R$. To describe solutions of the second condition we introduce the following conjugation on SU(2),

$$\gamma\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = \overline{\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

In the case of the vector gauging, $\epsilon_R = \epsilon_L$ and hence we look for pairs of Ω_1, Ω_2 such that $\Omega_2 \circ \epsilon \circ \Omega_1 = \epsilon$. This is satisfied for $(\Omega_1, \Omega_2) = (\mathrm{id}, \mathrm{id})$ and for $(\Omega_1, \Omega_2) = (\gamma, \gamma)$. While the first choice gives A-branes, the second is associated with B-branes in the terminology of [17]. The analysis of axial gauging is similar and leads to the two possibilities $(\Omega_1, \Omega_2) = (\mathrm{id}, \gamma)$ and $(\Omega_1, \Omega_2) = (\gamma, \mathrm{id})$.

To describe the D-branes in the parafermion theory we apply our general scheme according to which we have to consider products of twisted conjugacy classes

$$\mathcal{C}^{\mathrm{SU}(2)}_\mu(\Omega_2) \cdot \Omega_2 \circ \epsilon\big(\mathcal{C}^{\mathrm{U}(1)}_a(\Omega_1)\big) = \Big\{ g g_\mu \Omega_2(g^{-1}) \cdot \Omega_2 \circ \epsilon\big(h h_a \Omega_1(h^{-1})\big)\Big\}\,.$$

Here, $g_\mu \in \mathrm{SU}(2)$ and $h_a \in \mathrm{U}(1)$ are two fixed elements and $g \in \mathrm{SU}(2), h \in \mathrm{U}(1)$ are allowed to run over the whole groups. To make this more explicit let us restrict to the case of vector gauging. For the A-branes one can easily see that the relevant conjugacy classes have the form

$$\begin{aligned} \mathcal{C}_\mu^{\mathrm{SU}(2)}(\mathrm{id}) &= \big\{ \big(c \pm i\sqrt{r^2-c^2}, \sqrt{1-r^2}\, e^{i\phi_2} \big) \big| 0 \leq |c| \leq r \leq 1 \big\} \\ \epsilon\big(\mathcal{C}_a^{\mathrm{U}(1)}(\mathrm{id})\big) &= \big\{ (e^{ia}, 0) \big\} \,. \end{aligned}$$

The A-branes are then parametrized by

$$\mathcal{C}_\mu^{\mathrm{SU}(2)} \cdot \epsilon\big(\mathcal{C}_a^{\mathrm{U}(1)}\big) = \big\{ \big((c \pm i\sqrt{r^2-c^2}) e^{ia}, \sqrt{1-r^2}\, e^{i(\phi_2-a)} \big) \big| 0 \leq |c| \leq r \leq 1 \big\} \,.$$

It is now very easy to depict these branes in the figures 1 and 2. After vector gauging we recover one-dimensional branes stretching between two points on the boundary of the disc. For $c = 1$ they degenerate to a point-like object on the boundary.

In the case of B-branes, we use the following twisted conjugacy classes on SU(2) and U(1) in our construction

$$\begin{aligned} \mathcal{C}_\mu^{\mathrm{SU}(2)}(\gamma) &= \big\{ \big(r\, e^{i\phi_1}, (c \pm i\sqrt{1-r^2-c^2}) \big) \big| 0 \leq r \leq \sqrt{1-c^2} \leq 1 \big\} \\ \gamma \circ \epsilon\big(\mathcal{C}_a^{\mathrm{U}(1)}(\gamma)\big) &= \big\{ (e^{i\xi}, 0) \big| 0 \leq \xi \leq 2\pi \big\} \,. \end{aligned}$$

If we take the product the branes obviously fill the whole space in the figures 1 and 2 — but only for values $0 \leq r \leq \sqrt{1-c^2} \leq 1$. After vector gauging we thus recover the usual B-branes which are two-dimensional discs centered around the origin of the target space disc. For $c = 0$ they degenerate to a truly space filling brane. It is not difficult to work out the corresponding results for axial gauging.

4.2 T^{pq} spaces and the conifold

The spaces T^{pq} that we are about to analyze next are simple generalization of the space T^{11}. The latter is a close relative of the base of the conifold in which the RR-fluxes of the latter are replaced by a NSNS background field [3]. Our general theory provides a large class of boundary theories for this background, including branes that wrap one of the three-spheres in T^{11}. Related objects play an important role in the conifold geometry.

The T^{pq} spaces are defined to be quotients $\big(\mathrm{SU}(2)_{k_1} \times \mathrm{SU}(2)_{k_2}\big) / \,\mathrm{U}(1)_k$ where the U(1) subgroup acts by twisted conjugation, i.e. according to $(g_1, g_2) \mapsto \big(g_1 \epsilon_p(h^{-1}), \epsilon_q(h) g_2\big)$ where $\epsilon_p(\eta) = \epsilon(\eta^p)$ and ϵ is the usual embedding of U(1) into SU(2). We obtain this action from the choice

$$\epsilon_L = e \times \epsilon_q \,, \qquad \epsilon_R = \epsilon_p \times e \,.$$

If we parametrize the first factor SU(2) by (z_1, z_2) as before and similarly use (z_1', z_2') for the second factor, we realize that the action of $\eta = \exp(i\tau) \in \mathrm{U}(1)$ can be stated more explicitly by the formula

$$(z_1, z_2, z_1', z_2') \;\mapsto\; (e^{-ip\tau} z_1, e^{ip\tau} z_2, e^{iq\tau} z_1', e^{iq\tau} z_2') \,. \tag{4.2}$$

The corresponding gauged WZNW functional is free of anomalies provided that $k = k_1 p^2 = k_2 q^2$ (see condition (2.2)). Note that the resulting coset still has a $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry which is realized by $(g_1, g_2) \mapsto \big(h_1 g_1, g_2 h_2\big)$.

The geometry of the coset may be deduced from the action (4.2) by putting z_1 to a positive real number. This works fine except for $z_1 = 0$ where we have to gauge z_2 for instance. The resulting geometry is based on a product of a two-sphere times a three-sphere. Due to the non-trivial embeddings only part of the U(1) has to be used for the gauging and a detailed analysis yields

$$\mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1) = (S^2 \times S^3)/\mathbb{Z}_p\,.$$

Let us now have a look for the D-branes in this geometry. According to the general procedure we are instructed to select a chain of groups. Here we will work with a chain of length $N = 3$ consisting of $U_1 = \mathrm{U}(1)$, $U_2 = \mathrm{U}(1) \times \mathrm{U}(1)$ and $U_3 = \mathrm{SU}(2) \times \mathrm{SU}(2)$. We also pick embedding maps $\epsilon_1 : \mathrm{U}(1) \to \mathrm{U}(1) \times \mathrm{U}(1)$ defined by $\epsilon_1(\eta) = e \times \eta$ and $\epsilon_2 : \mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ given through $\epsilon_2 = \epsilon_p \times \epsilon_q$. Furthermore, we shall assume that the automorphism Ω_1 is the identity map. The other two automorphisms $\Omega_2 = \Omega'$ and $\Omega_3 = \Omega$ are allowed to be non-trivial. D-branes obtained from these data wrap the following product of twisted conjugacy classes,

$$\mathcal{C}^{\mathrm{SU}(2)\times\mathrm{SU}(2)}(\Omega) \cdot \Omega \circ \epsilon_L\big(\mathcal{C}^{\mathrm{U}(1)\times\mathrm{U}(1)}(\Omega')\big)\,.$$

In writing this formula, we omitted the factor associated with a conjugacy class in $U_1 = \mathrm{U}(1)$. Conjugacy classes of U(1) consist of a single point which we can choose to be the unit element. The sets above descend to the coset space T^{pq} if $\Omega \circ \epsilon_L \circ \Omega' = \epsilon_R$ (note that the condition $\epsilon_2 \circ \epsilon_1 = \epsilon_L$ holds by construction). One solution to this condition is $\Omega = \mathrm{id} \times \mathrm{id}$ and $\Omega' = \sigma$ the permutation of the two U(1) factors.

Given these gluing automorphisms, the relevant conjugacy classes in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ are given by

$$\mathcal{C}^{\mathrm{SU}(2)}_\mu \times \mathcal{C}^{\mathrm{SU}(2)}_\nu = \big\{\big(re^{\pm i\theta(r)}, \sqrt{1-r^2}e^{i\phi_2}, r'e^{\pm i\theta'(r')}, \sqrt{1-r'^2}e^{i\phi'_2}\big)\big|c_\mu \le r \le 1, c_\nu \le r' \le 1\big\}$$

where the signs $\pm$ may be chosen independently and $r\cos\theta(r) = c_\mu = \mathrm{const.}$, $r'\cos\theta'(r') = c_\nu = \mathrm{const.}$ These equations may only be solved for $r \ge c_\mu$ and $r' \ge c_\nu$. The twisted conjugacy classes in $\mathrm{U}(1) \times \mathrm{U}(1)$ are of the form

$$\mathcal{C}^{\mathrm{U}(1)\times\mathrm{U}(1)}_a(\Omega = \sigma) = \big\{\big(e^{i(a+\xi)}, e^{i(a-\xi)}\big)\big\} \overset{\epsilon_L}{\mapsto} \big\{\big(e^{i(a+\xi)}, 0, e^{i(a-\xi)}, 0\big)\big\}\,.$$

Combining these two results we find the following expression for the product

$$\begin{aligned}&\mathcal{C}^{\mathrm{SU}(2)}_\mu \times \mathcal{C}^{\mathrm{SU}(2)}_\nu \cdot \epsilon\big(\mathcal{C}^{\mathrm{U}(1)\times\mathrm{U}(1)}_a(\Omega = \sigma)\big) = \\ &\quad = \big\{\big(re^{\pm i\theta(r)+ip(a+\xi)}, \sqrt{1-r^2}e^{i(\phi_2-p(a+\xi))}, r'e^{\pm i\theta'(r')+iq(a-\xi)}, \sqrt{1-r'^2}e^{i(\phi'_2-q(a-\xi))}\big)\big\}\end{aligned}$$

We may use the gauge freedom to put the first entry to r. This is equivalent to setting $p\tau = \pm\theta(r) + p(a+\xi)$ in eq. (4.2). The resulting terms in the second and fourth entry may be compensated by a redefinition of ϕ_2 and ϕ_2'. We are thus left with

$$\mathcal{C}_\mu^{\mathrm{SU}(2)} \times \mathcal{C}_\nu^{\mathrm{SU}(2)} \cdot \epsilon\big(\mathcal{C}_a^{\mathrm{U}(1)\times\mathrm{U}(1)}(\Omega=\sigma)\big) / \,\mathrm{U}(1) = \\ = \Big\{ \big(r, \sqrt{1-r^2}e^{i\phi_2}, r' e^{\pm i\theta'(r') \pm iq/p\theta(r)+2iqa}, \sqrt{1-r'^2}e^{i\phi_2'}\big) \Big\} \,.$$

Let us point out that after gauging we eliminated the variable ξ that parametrized the twisted conjugacy classes in $\mathrm{U}(1) \times \mathrm{U}(1)$. Hence, these branes in T^{pq} have the same dimensionality as conjugacy classes in $\mathrm{SU}(2) \times \mathrm{SU}(2)$, i.e. they are $0, 2$ or $4-$dimensional.

We can also construct odd dimensional branes in T^{pq} but this requires to change some of the data we have been using. We stay with the same groups U_s and embeddings as above but choose a different collection of automorphisms. For the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ we use the non-trivial inner automorphism $\Omega_3 = (\gamma, \gamma)$ whose constituents have been defined in eq. (4.1). The condition (3.3) may then be fulfilled if the automorphism Ω_2 of $\mathrm{U}(1) \times \mathrm{U}(1)$ is given by the exchange of group factors and Ω_1 by the inversion γ.

Under these circumstances, the twisted conjugacy classes in the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ are typically four-dimensional submanifolds of the form $S^2 \times S^2$ while those of $\mathrm{U}(1) \times \mathrm{U}(1)$ and $\mathrm{U}(1)$ are both one-dimensional. One may easily see that the product of them inside $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is a submanifold of dimension 2, 4 or 6. After gauging the $\mathrm{U}(1)$ we are thus left with all kinds of odd-dimensional branes. When the levels are even, it is possible to find three-dimensional branes which fill one of the three-spheres of T^{pq}. Related objects play an important role for string theory on the conifold.

4.3 The big-bang big-crunch scenario

Recently, there has been renewed interest [6] in the Nappi-Witten background [5] which describes a closed universe between a big-bang and a big-crunch singularity. It was shown that the dynamics couples the closed universe to regions in space-time which formerly were believed to be unphysical. The full geometry is given by the coset $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)/\mathbb{R} \times \mathbb{R}$ where the groups in the numerator act asymmetrically on both factors in the denominator. Here we shall apply our general framework to the discussion of brane geometries in these asymmetric cosets. We believe that the construction of the corresponding boundary states in these non-compact backgrounds is possible using results from [20, 21].

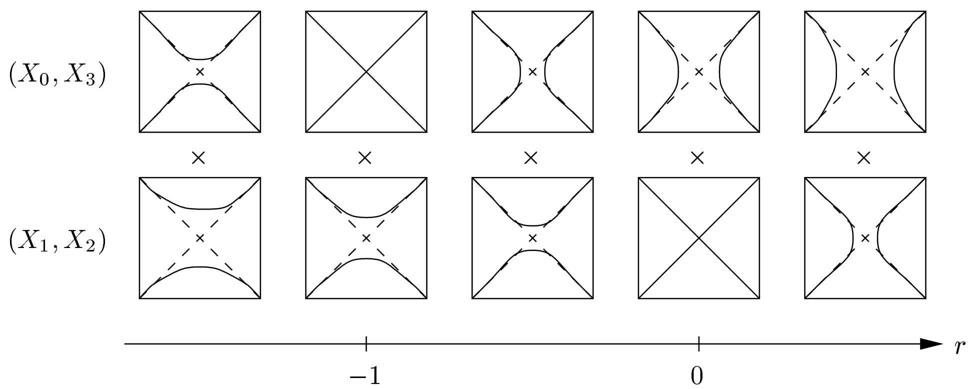

Figure 3: The group manifold $\mathrm{SL}(2,\mathbb{R})$.

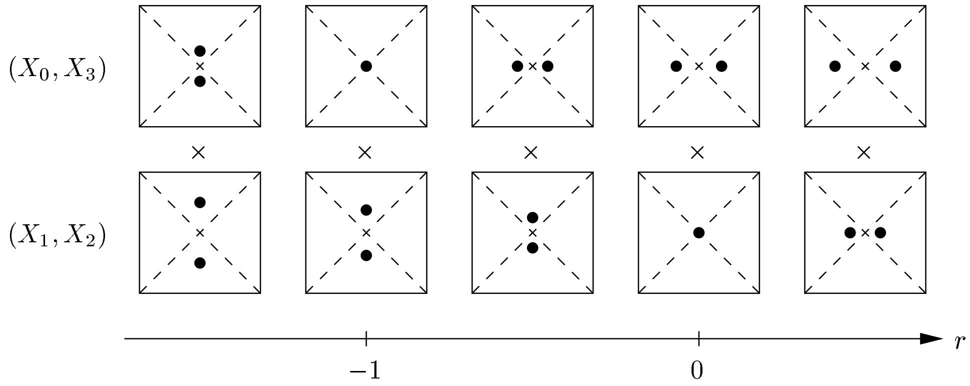

Figure 4: The group manifold $\mathrm{SL}(2,\mathbb{R})$ after gauging.

Let us review the geometry of the target space first. For our purposes it is convenient to parametrize the group manifold $\mathrm{SL}(2,\mathbb{R})$ according to

$$\begin{pmatrix} X_0 + X_3 & X_1 + X_2 \\ X_1 - X_2 & X_0 - X_3 \end{pmatrix} \qquad \text{with } X_0^2 - X_1^2 + X_2^2 - X_3^2 = 1\,, \quad X_i \in \mathbb{R}\,. \tag{4.3}$$

In close analogy to the case of $\mathrm{SU}(2)$, this set can be depicted as a product of hyperbolas $X_1^2 - X_2^2 = r$ and $X_0^2 - X_3^2 = 1 + r$ in the (X_1, X_2)-plane and the (X_0, X_3)-plane, respectively. These hyperbolas are fibered over the real coordinate r and they degenerate in one of the two planes for $r = -1, 0$. We thus have to distinguish the regions $r > 0$, $0 > r > -1$ and $-1 > r$. The resulting geometry is pictured in figure 3 as a fibre over $r \in \mathbb{R}$. The parametrization of $\mathrm{SU}(2)$ has already been given in section 4.1 (see figures 1 and 2).

In the next step we have to specify the action of the subgroup $\mathbb{R}\times\mathbb{R}$ on $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$. To make contact with the general setting of section 2 let us introduce the notation $G = G_1 \times G_2 = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ and $H = \mathbb{R} \times \mathbb{R}$. The coset we want to consider is defined by using the identification $g \sim \epsilon_L(h) g \epsilon_R(h^{-1})$ where the left and right homomorphisms of the subgroup H are defined by [5]

$$\begin{aligned} \epsilon_L(\rho,\tau) &= \begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix} \times \begin{pmatrix} e^{i\tau} & 0 \\ 0 & e^{-i\tau} \end{pmatrix} \\ \epsilon_R(\rho,\tau) &= \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix} \times \begin{pmatrix} e^{-i\rho} & 0 \\ 0 & e^{i\rho} \end{pmatrix} . \end{aligned}$$

Using these expressions it is not difficult to see that the action of H leaves the quantities $X_0^2 - X_3^2$, $X_1^2 - X_2^2$, $|z_1|$ and $|z_2|$ invariant. In fact, these transformations correspond to boosts on the hyperbolas and rotations on the circles. Deviating from the analysis in [6] we will perform the gauge fixing completely in the $\mathrm{SL}(2,\mathbb{R})$ part of the target space. As can easily be seen, the gauge transformations allow to gauge the $\mathrm{SL}(2,\mathbb{R})$ hyperbolas down to two disconnected points. This procedure completely removes the gauge freedom except for singular points at $r = -1, 0$. These points correspond to the big-bang and big-crunch singularities and we will not be concerned too much with details of the geometry at these special points. The findings of these considerations are illustrated in the figures 4 and 5.

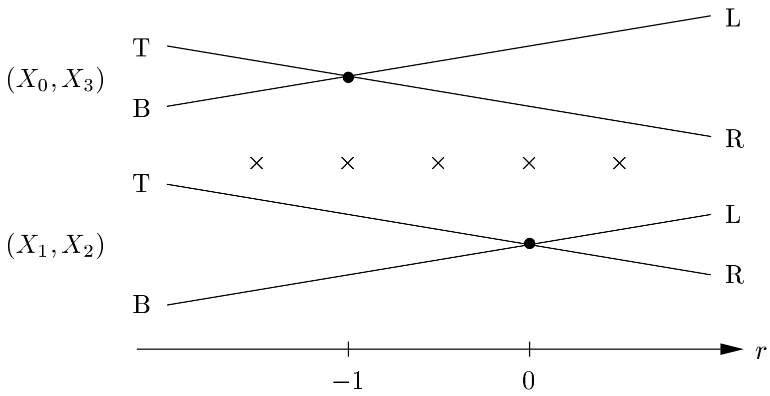

Figure 5: An alternative representation of the group manifold $\mathrm{SL}(2, \mathbb{R})$ after gauging.

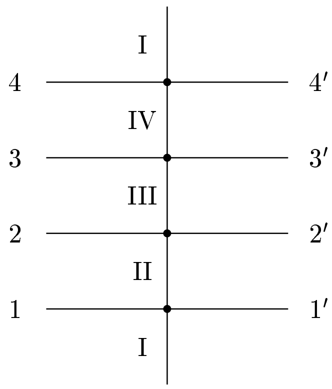

Figure 6: The big-bang/big-crunch scenario.

It is now only a short step to recover the results of [6]. Let us introduce the notation L, R, T, B which are shorthand for left, right, top and bottom and specify the location of points in figure 4. The regions of $\mathrm{SL}(2, \mathbb{R})$ which appear in the fibre over $r \in \mathbb{R}$ can be described by pairs of symbols L, R, T, B. A short look at figure 4 reveals that only twelve different combinations are allowed. Working out the connectivity properties of these different regions we arrive at figure 6 which has also been obtained in [6]. In order to simplify the comparison with [6] we have adopted their notation. The translation can be performed by means of table 1 (see also figure 7). From figure 6 we observe that there are four closed compact universes I–IV which are connected at the big-bang and big-crunch singularities. At each instant of time they have the topology of a three-sphere S^3 if one takes the $\mathrm{SU}(2)$ factor into account. The periodicity of time may be resolved by going to the infinite cover AdS_3 of $\mathrm{SL}(2, \mathbb{R})$. In addition to the closed universes there are eight whiskers which are also connected to the singularities. Over each point in the whisker one has a S^3.

Let us now begin to place branes into this geometry. Once more we shall work with chains of length $N = 2$ and the homomorphism $\epsilon = \epsilon_L : U_1 = \mathbb{R} \times \mathbb{R} \to U_2 = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SU}(2)$. If we define an automorphism Ω of $\mathbb{R} \times \mathbb{R}$ by $\Omega(\tau, \rho) = (-\rho, -\tau)$ then our condition (3.3) is satisfied whenever

$$\Omega_2 \circ \epsilon \circ \Omega_1 = \epsilon \circ \Omega \,.$$

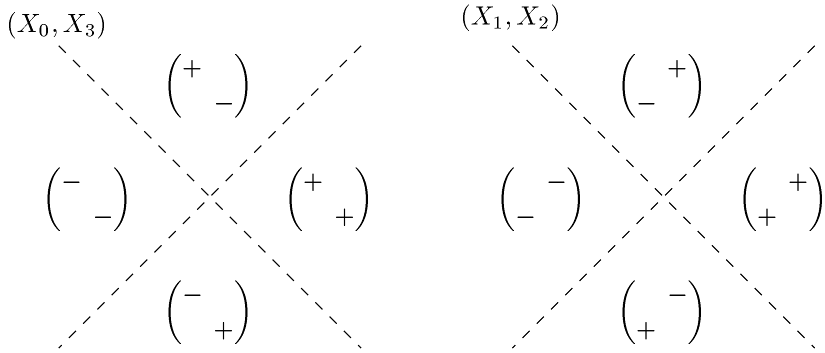

Figure 7: Different regions of $\mathrm{SL}(2,\mathbb{R})$ and where they appear in our picture. The matrix elements indicate the sign of $X_0 \pm X_3$ and $X_1 \pm X_2$, respectively.

D-branes in our background should be localized along the following product of twisted conjugacy classes,

$$\left[\mathcal{C}_\mu^{\mathrm{SL}(2,\mathbb{R})}(\omega_1) \times \mathcal{C}_\nu^{\mathrm{SU}(2)}(\omega_1')\right] \cdot (\omega_1 \times \omega_1') \circ \epsilon\big(\mathcal{C}_a^{\mathbb{R}\times\mathbb{R}}(\Omega_2)\big)\,, \tag{4.4}$$

before projecting to the coset. Here, we split $\Omega_1 = \omega_1 \times \omega_1'$ into the product of automorphisms for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$, respectively. There are several choices of automorphisms $\Omega_2, \omega_1, \omega_1'$ which satisfy our condition and we will discuss all of them in the following.

Let us start with the discussion of the twisted conjugacy class $\mathcal{C}_a^{\mathbb{R}\times\mathbb{R}}(\Omega_2)$. The most general automorphism of the additive group $\mathbb{R}\times\mathbb{R}$ is implemented by a non-singular 2×2-matrix. In our situation, however, not all choices are allowed. The only choices which have the chance to be consistent with condition (3.3) are $\Omega_2(\rho,\tau) = (\eta\tau, \xi\rho)$ where $\eta, \xi = \pm 1$. The resulting geometry is given by

$$\mathcal{C}_a^{\mathbb{R}\times\mathbb{R}}(\Omega_2) = \begin{cases} \mathbb{R}\times\mathbb{R}, & \text{for } \xi = -\eta \\ \big\{(f_1+\lambda, f_2-\eta\lambda)\,\big|\,\lambda\in\mathbb{R}\big\}\,, & \text{for } \xi = \eta\,. \end{cases} \tag{4.5}$$

The embedding of these sets into $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ via the map $(\omega_1 \times \omega_1') \circ \epsilon$ leads to the same result in both cases after gauge fixing.

When investigating the geometry of the D-branes (4.4) in the big-bang big-crunch target space it is convenient to focus on the $\mathrm{SL}(2,\mathbb{R})$ part as all interesting features arise from this factor. We thus only have to distinguish two different cases, corresponding to the two types of twisted conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$. As most of the group $\mathrm{SL}(2,\mathbb{R})$ will be gauged away, it suffices to address the following two questions:

1. Which ranges of r are covered by the twisted conjugacy classes?

2. Does the twisted conjugacy class extend along one or even both branches of the hyperbolas, i.e. does the D-brane cover one or two points for fixed value of r after gauging?

The twisted conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ are easily described. For untwisted conjugacy classes one has two types. There are two point-like conjugacy classes which correspond

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(R,B)	(R,T)	(L,T)	(L,B)	(R,R)	(R,L)	(B,T)	(T,T)	(L,L)	(L,R)	(T,B)	(B,B)
I	II	III	IV	1	$1'$	2	$2'$	3	$3'$	4	$4'$

Table 1: Translation table for the twelve different regions.

to the center of $\mathrm{SL}(2,\mathbb{R})$ while all others are two-dimensional. The exact shape has been worked out in [22, 23] but we will not need these details. The point-like branes are specified by $X_0 = \pm 1$ and $X_1 = X_2 = X_3 = 0$, i.e. they are localized at $r = 0$. After gauging they sit at the singularities between the closed universes I–II and III–IV, respectively. The two-dimensional conjugacy classes are of the form $X_0 = C = \text{const.}$ with arbitrary values of the remaining coordinates. According to the constraint (4.3) we obtain $r = C^2 - 1 - X_3^2 \leq C^2 - 1$. This means that the conjugacy class after gauging covers at least all four whiskers $2, 2', 4, 4'$. For $C \neq 0$ the conjugacy class grows into two of the four closed universes starting from the singularity which joins them. Depending on the sign of C these are the regions I–II (for $C > 0$) and III–IV (for $C < 0$). If $|C|$ reaches the value 1 (from below) the conjugacy class stretches completely through both of the closed universes. Increasing $|C|$ further, the conjugacy classes start to reach into two of the remaining whiskers — $1, 1'$ for $C > 1$ and $3, 3'$ for $C < -1$. Note, that the multiplication with the twisted conjugacy class of $\mathbb{R}\times\mathbb{R}$ has no influence on the possible values of r as it simply corresponds to some boost on the hyperbolas which will be gauged away in any case.

The twisted conjugacy classes arise from the automorphism which reverses the sign of X_2 and X_3. It may be described by conjugation with the element $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The corresponding twisted conjugacy classes are given by $\mathrm{tr}(Mg) = 2X_1 = 2C = \text{const.}$ According to the constraint (4.3) we obtain $r = C^2 - X_2^2 \leq C^2$. The discussion is similar as in the untwisted case. For all values of C the twisted conjugacy classes pass through all four closed universes I–IV and the four whiskers $2, 2', 4, 4'$. For $C \neq 0$ the conjugacy classes also cover part of the whiskers $1, 3'$ ($C > 0$) or $1', 3$ ($C < 0$). The results of the last two paragraphs are illustrated in figure 8.

So far we have only considered the $\mathrm{SL}(2,\mathbb{R})$ part of the target space. To obtain the complete picture we have also to take the $\mathrm{SU}(2)$ part into account as well as the product with the twisted conjugacy class $\mathcal{C}_a^{\mathbb{R}\times\mathbb{R}}(\omega)$. We already argued that the latter has no effect on the $\mathrm{SL}(2,\mathbb{R})$ part as it does not affect the value of r and may thus be gauged away. This statement also implies that the resulting D-branes factorize (in the same sense as the gauge fixing factorized). If we try to solve condition (3.3) with $\omega_1 = \mathrm{id}$, i.e. if we want to take the ordinary conjugacy classes in the $\mathrm{SL}(2,\mathbb{R})$ part, we have to use an automorphism Ω_2 of $\mathbb{R}\times\mathbb{R}$ with $\eta = 1$. Depending on the choice of ξ we are still able to obtain both expressions for twisted conjugacy classes that appear in eq. (4.5). The same statement holds true for $\eta = -1$, i.e. for the case of a twisted conjugacy class in the $\mathrm{SL}(2,\mathbb{R})$ part.

It is now very simple to describe the geometry of the D-branes in the $\mathrm{SU}(2)$ part. We simply have to multiply the (shifted) conjugacy class of $\mathrm{SU}(2)$ with elements of the form $\mathrm{diag}(e^{i\lambda}, e^{-i\lambda})$ for all values of λ. As was observed in [17, 24] and more in the spirit of our approach in [9], this corresponds to a smearing of the original conjugacy class.

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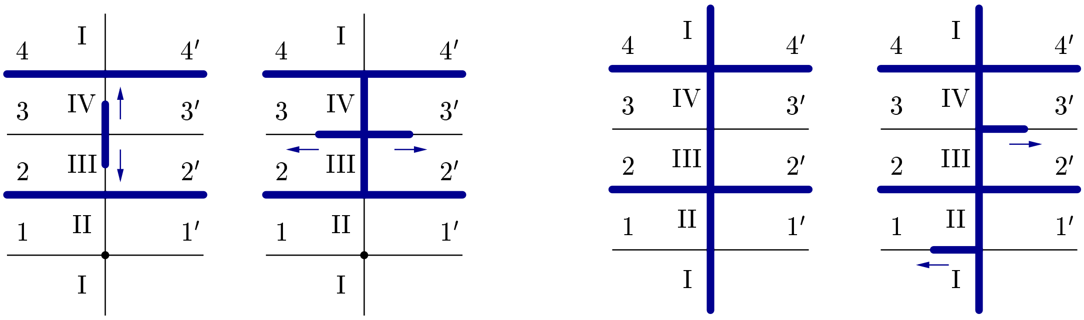

Figure 8: D-branes in the big-bang big-crunch scenario. The branes on the left hand side have been constructed with an ordinary conjugacy class of $SL(2,\mathbb{R})$ while for the right ones a twisted conjugacy class was employed.

Let us conclude with a short summary of our results. All essential information about the target space and about its D-branes are contained in the figure 8. While the D-branes cover the high-lightened regions in the $SL(2,\mathbb{R})$ part, we also have a three-sphere over each of these points which is partly covered by the D-brane. The geometry of the latter is either given by a circle around some equator or by a smeared two-sphere which covers a three-dimensional subset of S^3.

5. Conclusions

In this work we presented a comprehensive description of asymmetric coset models G/H for which the action of H on G is not necessarily given by the adjoint. A bulk partition function was proposed based on a semi-classical analysis in the large volume limit and the modular invariance of this partition function was shown to be equivalent to the anomaly cancellation that is known from the lagrangian description of the coset.

We then provided a general prescription of constructing branes in asymmetric cosets. Due to the heterotic nature of the models, one is naturally lead to boundary states which break part of the symmetry of the bulk theory. The geometry of the branes may be deduced from those of symmetry breaking branes on group manifolds [9]. Branes which possess a symmetry compatible with the gauge action were argued to descend naturally to the coset.

Our general findings have been used to construct D-branes in the cosmological Nappi-Witten background (big-bang big-crunch space-time) and in the base $S^2 \times S^3$ of the conifold. Among the branes in the big-bang big-crunch space-time there are examples which cross the singularities and run through all the universes. In the base of the conifold we found branes of all dimensions. For even values of the level one may construct branes which fill one of the three-spheres.

Before we conclude let us mention a few open problems which remain to be solved. Our algebraic construction of branes in asymmetric cosets was designed for the case that left and right action of the gauge group can be related by automorphisms in a chain of intermediate groups (asymmetric cosets of generalized automorphism type). If this condition is not fulfilled, one could be tempted to follow [7] (see also [9]) and to propose generalized conjugacy classes for the geometry of the corresponding branes. While this procedure works fine on the lagrangian level, we have not been able to implement it in an algebraic approach.

Another issue is the discussion of the stability and the dynamics of our branes. In the geometric regime the question of stability should be accessible from a Born-Infeld analysis [17]. One may even hope to go one step beyond such an analysis and to construct the non-commutative gauge theory which governs the dynamics of branes in asymmetric cosets [25]. It may also well be that the results of [26] generalize to asymmetric cosets.

Acknowledgments

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A. The GMM cosets revisited

Models of the type which have been discussed in section 2.4.2 first appeared in [1, 2]. These authors presented a lagrangian formulation of these theories and considered the associated current algebra. In our opinion their discussion of the algebraic properties is not completely accurate. In particular they argued that the energy momentum tensor is *not* obtained by the standard affine Sugawara [27, 28, 29] and coset constructions [29, 30] which seems to be incorrect. We take this as an opportunity to review the lagrangian description and to clarify some statements.

The gauged WZNW functional (2.4) is quadratic in the gauge fields. It may thus be simplified — in principle — by integrating out the gauge fields. The resulting expressions will, however, remain quite formal in the general case (see however [31, 32, 33, 34, 35, 36, 37]). The reason for these difficulties is the third term in the interaction functional (2.5) which does not only contain the gauge fields A and $\bar{A}$ but also the group element g. For the gaussian path integral to be performed one would need to diagonalize the quadratic form matrix which depends explicitly on g.

For our particular choice of embeddings the corresponding term vanishes and the path integral may easily be evaluated. The interaction functional (2.5) reduces to

$$\begin{aligned} S_{\text{int}}^{G_1\times G_2/H}(g_1,g_2,A,\bar{A}|k,\epsilon_{L/R}) =& \frac{k_1}{4\pi}\int_\Sigma d^2z\ \operatorname{Tr}_1\big\{-2\epsilon_1(A)g_1^{-1}\bar{\partial}g_1 - \epsilon_1(\bar{A})\epsilon_1(A)\big\} \\ &+ \frac{k_2}{4\pi}\int_\Sigma d^2z\ \operatorname{Tr}_2\big\{2\epsilon_2(\bar{A})\partial g_2 g_2^{-1} - \epsilon_2(\bar{A})\epsilon_2(A)\big\}\,. \end{aligned}$$

It is fairly simple to read off the quadratic form matrix from this expression and integrate out the gauge fields in full generality. We only have to be a bit careful about our notations. We may decompose the $\mathfrak{h}$-valued gauge fields A and $\bar{A}$ according to $A = A_\alpha T^\alpha$ and $\bar{A} =$

$\bar{A}_\alpha T^\alpha$. The *abstract* Lie algebra generators satisfy the commutation relations $[T^\alpha, T^\beta] = if^{\alpha\beta}{}_\gamma T^\gamma$. Indices are raised and lowered using the Killing form[5]

$$2\,\kappa^{\alpha\beta} = \mathrm{Tr}\big\{T^\alpha T^\beta\big\}$$

and its inverse. We may choose generators $T^i \in \{\epsilon_1(T^\alpha), T^I\}$ of $\mathfrak{g}_1$ and generators $T^a \in \{\epsilon_2(T^\alpha), T^A\}$ of $\mathfrak{g}_2$. These satisfy $[T^i, T^j] = if^{ij}{}_k T^k$ and $[T^a, T^b] = if^{ab}{}_c T^c$. If all three indices take values in the subalgebra $\mathfrak{h}$, the structure constants by construction just reduce to the structure constants of $\mathfrak{h}$ in the given basis. This is only true as long as the index structure is as indicated because one would have to use different Killing forms to lower the indices. From (2.1) it follows that they satisfy

$$\kappa^{\alpha\beta} = \frac{\kappa_1^{\alpha\beta}}{x_1} = \frac{\kappa_2^{\alpha\beta}}{x_2}\,.$$

We see the embedding indices x_i entering this expression.

The last relations imply

$$\mathrm{Tr}_1\big\{\epsilon_1(\bar{A})\,\epsilon_1(A)\big\} = 2\,x_1\,\bar{A}_\alpha A^\alpha \qquad \mathrm{Tr}_2\big\{\epsilon_2(\bar{A})\,\epsilon_2(A)\big\} = 2\,x_2\,\bar{A}_\alpha A^\alpha\,.$$

The formula

$$\int d^n y\; e^{-\frac{1}{2}y^T \mathbb{A} y + b^T y} = \frac{(2\pi)^{n/2}}{\sqrt{\det \mathbb{A}}}\; e^{\frac{1}{2}\,b^T \mathbb{A}^{-1} b}\,.$$

for the gaussian path integral may thus be applied with

$$y = \begin{pmatrix} A^\alpha \\ \bar{A}^\beta \end{pmatrix} \qquad \mathbb{A} = \begin{pmatrix} 0 & \frac{x_1 k_1}{\pi}\,\kappa_{\alpha\beta} \\ \frac{x_2 k_2}{\pi}\,\kappa_{\alpha\beta} & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -\frac{k_1}{2\pi}\,\mathrm{Tr}_1\big\{\epsilon_1(T^\alpha) g_1^{-1}\bar{\partial} g_1\big\} \\ \frac{k_2}{2\pi}\,\mathrm{Tr}_2\big\{\epsilon_2(T^\alpha)\partial g_2 g_2^{-1}\big\} \end{pmatrix}\,.$$

The matrix $\mathbb{A}$ is symmetric as by assumption $k = x_1 k_1 = x_2 k_2$. It may easily be inverted. After performing the gaussian path integral the interaction term reads

$$S_{\text{int}}^{G_1\times G_2/H}(g_1, g_2|\,k, \epsilon_{L/R}) = -\frac{\kappa_{\alpha\beta}\,k_1 k_2}{4\pi\,k}\int_\Sigma d^2 z\;\mathrm{Tr}_1\big\{\epsilon_1(T^\alpha)\,g_1^{-1}\bar{\partial} g_1\big\}\;\mathrm{Tr}_2\big\{\epsilon_2(T^\beta)\,\partial g_2 g_2^{-1}\big\} \tag{A.1}$$

This is exactly the action functional which was constructed in [1, 2].

The action functional (A.1) possesses a number of very interesting and useful symmetries. By construction it is invariant under the infinitesimal gauge transformations $(g_1, g_2) \mapsto \big(g_1(1 - i\,\epsilon_1(\Omega)), (1 + i\,\epsilon_2(\Omega))g_2\big)$ with $\Omega = \Omega(z, \bar{z}) \in \mathfrak{h}$. In addition, the model admits the symmetry $G_1^L(z) \times G_2^R(\bar{z})$, i.e. it is invariant under $(g_1, g_2) \mapsto \big(g_1'(z) g_1, g_2 g_2'^{-1}(\bar{z})\big)$. The last symmetry is generated by the currents

$$\begin{aligned} J(z) &= J_1 - \frac{k_2\,\kappa_{\alpha\beta}}{2\,x_1}\; g_1 \epsilon_1(T^\alpha) g_1^{-1}\;\mathrm{Tr}_2\big\{\epsilon_2(T^\beta)\,\partial g_2 g_2^{-1}\big\} \\ \bar{J}(\bar{z}) &= \bar{J}_2 + \frac{k_1\,\kappa_{\alpha\beta}}{2\,x_2}\; g_2^{-1}\epsilon_2(T^\beta) g_2\;\mathrm{Tr}_1\big\{\epsilon_1(T^\alpha)\,g_1^{-1}\bar{\partial} g_1\big\}\,. \end{aligned}$$

[5]We remind the reader that Tr is a normalized trace and that we work in the conventions of [11].

They satisfy $\bar\partial J(z) = \partial \bar J(\bar z) = 0$ by the equations of motion. During the derivation we used the relation $x_1 k_1 = x_2 k_2$. Note that J takes values in the Lie algebra $\mathfrak{g}_1$, while $\bar J$ is from $\mathfrak{g}_2$. This means that the index structure is J^i, $\bar J^a$ which makes explicit the heterotic nature of our coset. Both currents are gauge invariant.

In the algebraic description of our asymmetric coset model we already assumed some properties which would have been expected from a straightforward generalization of the GKO construction. We are now able to justify this procedure more rigorously by working out the energy momentum tensor and the commutation relations of the currents. Let us start with the latter. It is convenient to introduce the fields

$$
\begin{aligned}
J_1 &= -k_1\, \partial g_1 g_1^{-1} \qquad & \bar J_1 &= k_1\, g_1^{-1} \bar\partial g_1 \\
J_2 &= -k_2\, \partial g_2 g_2^{-1} \qquad & \bar J_2 &= k_2\, g_2^{-1} \bar\partial g_2
\end{aligned}
$$

which correspond to the (former) G_1 and G_2 currents, respectively. In terms of these quantities one obtains

$$
\begin{aligned}
J(z) &= J_1 + \frac{\kappa_{\alpha\beta}}{2x_1}\, g_1 \epsilon_1(T^\alpha) g_1^{-1}\ \mathrm{Tr}_2\{\epsilon_2(T^\beta) J_2\} \\
\bar J(\bar z) &= \bar J_2 + \frac{\kappa_{\alpha\beta}}{2x_2}\, g_2^{-1} \epsilon_2(T^\beta) g_2\ \mathrm{Tr}_1\{\epsilon_1(T^\alpha) \bar J_1\}\,.
\end{aligned}
$$

The symmetry $G_1^L(z) \times G_2^R(\bar z)$ implies the Ward identities [38, (15.40)]

$$
\begin{aligned}
\delta_L^{(1)} \langle X(w,\bar w)\rangle &= -\oint \frac{dz}{2\pi i}\, \Omega_i \langle J^i(z) X(w,\bar w)\rangle \\
\delta_R^{(2)} \langle X(w,\bar w)\rangle &= \oint \frac{d\bar z}{2\pi i}\, \Omega_a \langle \bar J^a(\bar z) X(w,\bar w)\rangle\,,
\end{aligned}
$$

which are related to the transformations $\delta_L^{(1)} g_1 = i\, \Omega_i T^i\, g_1$ and $\delta_R^{(2)} g_2 = -i\, g_2\, \Omega_a T^a$. From the previous equations we may derive the non-trivial OPEs

$$
\begin{aligned}
J^i(z) J^j(w) &= \frac{i f^{ij}{}_k}{z-w}\ J^k(w) + \frac{k_1\, \kappa_1^{ij}}{(z-w)^2} \\
J^i(z) g_1(w,\bar w) &= -\frac{T^i g_1(w,\bar w)}{z-w} \\
J^i(z) J_1^j(w,\bar w) &= \frac{i f^{ij}{}_k}{z-w}\ J_1^k(w,\bar w) + \frac{k_1\, \kappa_1^{ij}}{(z-w)^2}\,.
\end{aligned}
$$

and

$$
\begin{aligned}
\bar J^a(\bar z) \bar J^b(\bar w) &= \frac{i f^{ab}{}_c}{\bar z-\bar w}\, \bar J^c(\bar w) + \frac{k_2\, \kappa_2^{ab}}{(\bar z-\bar w)^2} \\
\bar J^a(\bar z) g_2(w,\bar w) &= \frac{g_2(w,\bar w) T^a}{\bar z-\bar w} \\
\bar J^a(\bar z) \bar J_2^b(w,\bar w) &= \frac{i f^{ab}{}_c}{\bar z-\bar w}\, \bar J_2^c(w,\bar w) + \frac{k_2\, \kappa_2^{ab}}{(\bar z-\bar w)^2}\,.
\end{aligned}
$$

All the other OPEs vanish:

$$\begin{aligned} J^i(z)\bar{J}^a(\bar{w}) &= J^i(z)g_2(w,\bar{w}) = J^i(z)\bar{J}_1^j(w,\bar{w}) = J^i(z)J_2^a(w,\bar{w}) = J^i(z)\bar{J}_2^a(w,\bar{w}) = 0 \\ \bar{J}^a(\bar{z})g_1(w,\bar{w}) &= \bar{J}^a(\bar{z})J_1^i(w,\bar{w}) = \bar{J}^a(\bar{z})\bar{J}_1^i(w,\bar{w}) = \bar{J}^a(\bar{z})J_2^b(w,\bar{w}) = 0\,. \end{aligned}$$

Let us emphasize the asymmetry in the OPEs which already showed up in the algebraic construction.

Now, that the current symmetry is under control we can focus our attention to the conformal symmetry, i.e. to the energy momentum tensor. Our treatment will reveal the central charge to be given by a combination of affine Sugawara and coset construction. Both left and right moving central charge agree. Due to the structure of the action functional for the asymmetric coset the *classical* chiral energy momentum tensors are given by

$$T = T_1 + T_2 + T_{\rm int} \qquad \text{and} \qquad \bar{T} = \bar{T}_1 + \bar{T}_2 + \bar{T}_{\rm int}\,.$$

The first two summands are the standard WZNW energy momentum tensors

$$\begin{aligned} T_1 &= \frac{1}{4\,k_1}\ {\rm Tr}_1\big\{J_1J_1\big\} & \bar{T}_1 &= \frac{1}{4\,k_1}\ {\rm Tr}_1\big\{\bar{J}_1\bar{J}_1\big\} \\ T_2 &= \frac{1}{4\,k_2}\ {\rm Tr}_2\big\{J_2J_2\big\} & \bar{T}_2 &= \frac{1}{4\,k_2}\ {\rm Tr}_2\big\{\bar{J}_2\bar{J}_2\big\}\,. \end{aligned}$$

The extra summands are given by

$$\begin{aligned} T_{\rm int} &= \frac{k_1\ \kappa_{\alpha\beta}}{4\,x_2}\ {\rm Tr}_1\big\{\epsilon_1(T^\alpha)\,g_1^{-1}\partial g_1\big\}\ {\rm Tr}_2\big\{\epsilon_2(T^\beta)\,\partial g_2 g_2^{-1}\big\} \\ \bar{T}_{\rm int} &= \frac{k_1\ \kappa_{\alpha\beta}}{4\,x_2}\ {\rm Tr}_1\big\{\epsilon_1(T^\alpha)\,g_1^{-1}\bar{\partial} g_1\big\}\ {\rm Tr}_2\big\{\epsilon_2(T^\beta)\,\bar{\partial} g_2 g_2^{-1}\big\}\,. \end{aligned}$$

It is very instructive to evaluate the expressions ${\rm Tr}_{R_1}\,JJ$ and ${\rm Tr}_{R_2}\,\bar{J}\bar{J}$. One is then naturally lead to

$$\begin{aligned} T &= \frac{1}{2k_1}\ J_iJ^i + \frac{1}{2k_2}\ (J_2)_a(J_2)^a - \frac{1}{2x_2k_2}\ (J_2)_\alpha(J_2)^\alpha = T_{k_1}^{G_1} + T_{k_2}^{G_2} - T_{x_2k_2}^{H} \\ \bar{T} &= \frac{1}{2k_2}\ \bar{J}_a\bar{J}^a + \frac{1}{2k_1}\ (\bar{J}_1)_i(\bar{J}_1)^i - \frac{1}{2x_1k_1}\ (\bar{J}_1)_\alpha(\bar{J}_1)^\alpha = \bar{T}_{k_1}^{G_1} + \bar{T}_{k_2}^{G_2} - \bar{T}_{x_1k_1}^{H}\,. \end{aligned}$$

The additional factors x_1 and x_2 arise due to the usage of the natural Killing form for $\mathfrak{h}$-quantities. After quantizing the theory the levels get shifted by the respective dual Coxeter numbers. Let us emphasize the following remarkable fact: Due to the condition $x_1k_1 = x_2k_2$ left and right moving Virasoro algebra possess the same central charge. This result also has been noted in [1, 2] but the algebraic reasons remained unclear. In particular in the last reference due to usage of intransparent notation it was not realized that the energy momentum tensor is actually defined by the standard affine Sugawara construction combined with the coset construction.

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